FINAL JEE–MAIN EXAMINATION – JANUARY, 2024

(Held On Monday 29th January, 2024) **TIME : 9 : 00 AM to 12 : 00 NOON**

MATHEMATICS | TEST PAPER WITH SOLUTION

SECTION-A

1. If in a G.P. of 64 terms, the sum of all the terms is 7 times the sum of the odd terms of the G.P, then the common ratio of the G.P. is equal to

(1) 7 (2) 4

(3) 5 (4) 6

Ans. (4)

Sol. $a+ar+ar^2+ar^3+....+ar^{63}$

$=7(a+ar^2+ar^4.....+ar^{62})$

$$\Rightarrow \frac{a(1-r^{64})}{1-r}=\frac{7a(1-r^{64})}{1-r^2}$$

$r = 6$

2. In an A.P., the sixth terms $a_6 = 2$. If the $a_1a_4a_5$ is the greatest, then the common difference of the A.P., is equal to

(1) $\frac{3}{2}$ (2) $\frac{8}{5}$ (3) $\frac{2}{3}$ (4) $\frac{5}{8}$

Ans. (2)

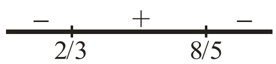
Sol. $a_6=2\Rightarrow a+5d=2$

$a_1a_4a_5=a(a+3d)(a+4d)$

$=(2-5d)(2-2d)(2-d)$

$f(d)=8-32d+34d^2-20d+30d^2-10d^3$

$f'(d)=-2(5d-8)(3d-2)$



$$d=\frac{8}{5}$$

3. If $f(x)=\begin{cases}2+2x, & -1\le x<0\\ 1-\frac{x}{3}, & 0\le x\le 3\end{cases}$; $g(x)=\begin{cases}-x, & -3\le x\le 0\\ x, & 0<x\le 1\end{cases}$,

then range of (fog(x)) is

(1) (0, 1] (2) [0, 3)

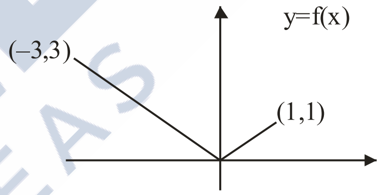
(3) [0, 1] (4) [0, 1)

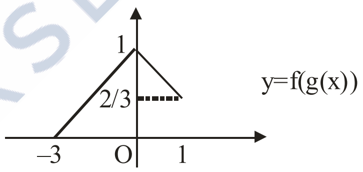
Ans. (3)

Sol. $f(g(x))=\begin{cases}2+2g(x), & -1\le g(x)<0 \quad .....(1)\\ 1-\frac{g(x)}{3}, & 0\le g(x)\le 3 \quad .....(2)\end{cases}$

By (1) $x\in\phi$

And by (2) $x\in[-3,0]$ and $x\in[0,1]$





Range of $f(g(x))$ is [0, 1]

4. A fair die is thrown until 2 appears. Then the probability, that 2 appears in even number of throws , is

(1) $\frac{5}{6}$ (2) $\frac{1}{6}$ (3) $\frac{5}{11}$ (4) $\frac{6}{11}$

Ans. (3)

Sol. Required probability =

$$\frac{5}{6}\times\frac{1}{6}+\left(\frac{5}{6}\right)^3\times\frac{1}{6}+\left(\frac{5}{6}\right)^5\times\frac{1}{6}+.....$$

$$=\frac{1}{6}\times\frac{\frac{5}{6}}{1-\frac{25}{36}}=\frac{5}{11}$$

5. If $z=\frac{1}{2}-2i$, is such that

$|z+1|=\alpha z+\beta(1+i), i=\sqrt{-1}$ and $\alpha,\beta\in R$, then

$\alpha+\beta$ is equal to

(1) –4 (2) 3

(3) 2 (4) –1

Ans. (2)

Sol. $z=\frac{1}{2}-2i$

$$|z+1|=\alpha z+\beta(1+i)$$

$$\left|\frac{3}{2}-2i\right|=\frac{\alpha}{2}-2\alpha i+\beta+\beta i$$

$$\left|\frac{3}{2}-2i\right|=\left(\frac{\alpha}{2}+\beta\right)+(\beta-2\alpha)i$$

$$\beta=2\alpha \text{ and } \frac{\alpha}{2}+\beta=\sqrt{\frac{9}{4}+4}$$

$$\alpha+\beta=3$$

6. $\lim\limits_{x\to\frac{\pi}{2}}\left(\frac{1}{\left(x-\frac{\pi}{2}\right)^2}\int_{x^3}^{\left(\frac{\pi}{2}\right)^3}\cos\left(\frac{1}{t^3}\right)dt\right)$ is equal to

(1) $\frac{3\pi}{8}$ (2) $\frac{3\pi^2}{4}$

(3) $\frac{3\pi^2}{8}$ (4) $\frac{3\pi}{4}$

Ans. (3)

Sol. Using L'hopital rule

$$=\lim_{x\to\frac{\pi}{2}^-}\frac{0-\cos x\times 3x^2}{2\left(x-\frac{\pi}{2}\right)}$$

$$=\lim_{x\to\frac{\pi}{2}^-}\frac{\sin\left(x-\frac{\pi}{2}\right)}{2\left(x-\frac{\pi}{2}\right)}\times\frac{3\pi^2}{4}$$

$$=\frac{3\pi^2}{8}$$

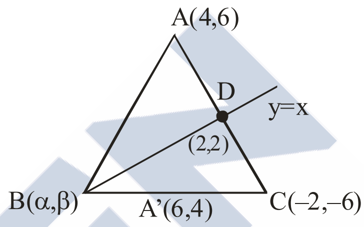
7. In a ΔABC, suppose $y = x$ is the equation of the bisector of the angle B and the equation of the side AC is $2x - y = 2$. If $2AB = BC$ and the point A and B are respectively (4, 6) and (α,β), then $\alpha+2\beta$ is equal to

(1) 42 (2) 39

(3) 48 (4) 45

Ans. (1)

Sol.



AD : DC = 1 : 2

$$\frac{4-\alpha}{6-\alpha}=\frac{10}{8}$$

$$\alpha=\beta$$

$$\alpha=14 \text{ and } \beta=14$$

8. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three non-zero vectors such that $\vec{b}$ and $\vec{c}$ are non-collinear .if $\vec{a}+5\vec{b}$ is collinear with $\vec{c}$, $\vec{b}+6\vec{c}$ is collinear with $\vec{a}$ and $\vec{a}+\alpha\vec{b}+\beta\vec{c}=\vec{0}$, then $\alpha+\beta$ is equal to

(1) 35 (2) 30

(3) – 30 (4)–25

Ans. (1)

Sol. $\vec{a}+5\vec{b}=\lambda\vec{c}$

$\vec{b}+6\vec{c}=\mu\vec{a}$

Eliminating $\vec{a}$

$$\lambda\vec{c}-5\vec{b}=\frac{6}{\mu}\vec{c}+\frac{1}{\mu}\vec{b}$$

$$\therefore \mu=\frac{-1}{5}, \lambda=-30$$

$$\alpha=5, \beta=30$$

9. Let $\left(5,\frac{a}{4}\right)$, be the circumcenter of a triangle with vertices $A(a,-2), B(a, 6)$ and $C\left(\frac{a}{4},-2\right)$. Let α denote the circumradius, β denote the area and γ denote the perimeter of the triangle. Then $\alpha+\beta+\gamma$ is

(1) 60 (2) 53
(3) 62 (4) 30

Ans. (2)

Sol. $A(a, -2), B(a, 6), C\left(\frac{a}{4},-2\right)$, $O\left(5,\frac{a}{4}\right)$

AO = BO

$$(a-5)^2+\left(\frac{a}{4}+2\right)^2=(a-5)^2+\left(\frac{a}{4}-6\right)^2$$

a = 8

AB = 8, AC = 6, BC = 10

$\alpha = 5, \beta = 24, \gamma = 24$

10. For $x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, if

$$y(x)=\int\frac{\operatorname{cosec}x+\sin x}{\operatorname{cosec}x\sec x+\tan x\sin^2 x}dx \text{ and}$$

$\lim\limits_{x\to\left(\frac{\pi}{2}\right)^-} y(x)=0$ then $y\left(\frac{\pi}{4}\right)$ is equal to

(1) $\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (2) $\frac{1}{2}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$

(3) $-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (4) $\frac{1}{\sqrt{2}}\tan^{-1}\left(-\frac{1}{2}\right)$

Ans. (4)

Sol. $y(x)=\int\frac{(1+\sin^2 x)\cos x}{1+\sin^4 x}dx$

Put sinx = t

$$=\int\frac{1+t^2}{t^4+1}dt=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t-\frac{1}{t}}{\sqrt{2}}\right)+C$$

$x=\frac{\pi}{2}, t=1$ $\therefore$ C = 0

$$y\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}\tan^{-1}\left(-\frac{1}{2}\right)$$

11. If $\alpha, -\frac{\pi}{2}<\alpha<\frac{\pi}{2}$ is the solution of $4\cos\theta+5\sin\theta=1$, then the value of $\tan\alpha$ is

(1) $\frac{10-\sqrt{10}}{6}$ (2) $\frac{10-\sqrt{10}}{12}$

(3) $\frac{\sqrt{10}-10}{12}$ (4) $\frac{\sqrt{10}-10}{6}$

Ans. (3)

Sol. $4+5\tan\theta=\sec\theta$

Squaring : $24\tan^2\theta+40\tan\theta+15=0$

$$\tan\theta=\frac{-10\pm\sqrt{10}}{12}$$

and $\tan\theta=-\left(\frac{10+\sqrt{10}}{12}\right)$ is Rejected.

(3) is correct.

12. A function y = f(x) satisfies

$f(x)\sin 2x+\sin x-(1+\cos^2 x)f'(x)=0$ with condition f(0) = 0 . Then $f\left(\frac{\pi}{2}\right)$ is equal to

(1) 1 (2) 0 (3) –1 (4) 2

Ans. (1)

Sol. $\frac{dy}{dx}-\left(\frac{\sin 2x}{1+\cos^2 x}\right)y=\sin x$

I.F. = $1 + \cos^2x$

$y\cdot(1+\cos^2 x)=\int(\sin x)dx$

= – cosx + C

x = 0, C = 1

$y\left(\frac{\pi}{2}\right)=1$

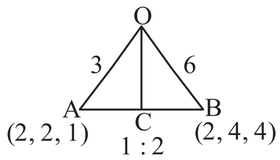
13. Let O be the origin and the position vector of A and B be $2\hat{i}+2\hat{j}+\hat{k}$ and $2\hat{i}+4\hat{j}+4\hat{k}$ respectively. If the internal bisector of $\angle AOB$ meets the line AB at C, then the length of OC is

(1) $\frac{2}{3}\sqrt{31}$ (2) $\frac{2}{3}\sqrt{34}$

(3) $\frac{3}{4}\sqrt{34}$ (4) $\frac{3}{2}\sqrt{31}$

Ans. (2)

Sol.



length of $OC = \frac{\sqrt{136}}{3} = \frac{2\sqrt{34}}{3}$

14. Consider the function $f : \left[\frac{1}{2}, 1\right] \to R$ defined by $f(x) = 4\sqrt{2}x^3 - 3\sqrt{2}x - 1$. Consider the statements

(I) The curve y = f(x) intersects the x-axis exactly at one point

(II) The curve y = f(x) intersects the x-axis at $x = \cos\frac{\pi}{12}$

Then

(1) Only (II) is correct

(2) Both (I) and (II) are incorrect

(3) Only (I) is correct

(4) Both (I) and (II) are correct

Ans. (4)

Sol. $f'(x) = 12\sqrt{2}x^2 - 3\sqrt{2} \geq 0$ for $\left[\frac{1}{2}, 1\right]$

$f\left(\frac{1}{2}\right) < 0$

$f(1) > 0 \Rightarrow$ (A) is correct.

$f(x) = \sqrt{2}\left(4x^3 - 3x\right) - 1 = 0$

Let $\cos\alpha = x$,

$\cos 3\alpha = \cos\frac{\pi}{4} \Rightarrow \alpha = \frac{\pi}{12}$

$x = \cos\frac{\pi}{12}$

(4) is correct.

15. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & \beta & \alpha \end{bmatrix}$ and $|2A|^3 = 2^{21}$ where $\alpha, \beta \in Z$,

Then a value of α is

(1) 3 (2) 5

(3) 17 (4) 9

Ans. (2)

Sol. $|A| = \alpha^2 - \beta^2$

$|2A|^3 = 2^{21} \Rightarrow |A| = 2^4$

$\alpha^2 - \beta^2 = 16$

$(\alpha + \beta)(\alpha - \beta) = 16 \Rightarrow \alpha = 4$ or 5

16. Let PQR be a triangle with $R(-1, 4, 2)$. Suppose M(2, 1, 2) is the mid point of PQ. The distance of the centroid of ΔPQR from the point of intersection of the line $\frac{x-2}{0} = \frac{y}{2} = \frac{z+3}{-1}$ and $\frac{x-1}{1} = \frac{y+3}{-3} = \frac{z+1}{1}$ is

(1) 69 (2) 9

(3) $\sqrt{69}$ (4) $\sqrt{99}$

Ans. (3)

Sol. Centroid G divides MR in 1 : 2

G(1, 2, 2)

Point of intersection A of given lines is (2,–6, 0)

$AG = \sqrt{69}$

17. Let R be a relation on Z × Z defined by

(a, b)R(c, d) if and only if ad – bc is divisible by 5.

Then R is

(1) Reflexive and symmetric but not transitive

(2) Reflexive but neither symmetric not transitive

(3) Reflexive, symmetric and transitive

(4) Reflexive and transitive but not symmetric

Ans. (1)

Sol. (a, b)R(a, b) as ab – ab = 0

Therefore reflexive

Let (a,b)R(c,d)⇒ad – bc is divisible by 5

⇒ bc – ad is divisible by 5 ⇒(c,d)R(a,b)

Therefore symmetric

Relation not transitive as (3,1)R(10,5) and (10,5)R(1,1) but (3,1) is not related to (1,1)

18. If the value of the integral

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{x^2\cos x}{1+\pi^x}+\frac{1+\sin^2 x}{1+e^{\sin x^{2023}}}\right)dx=\frac{\pi}{4}(\pi+a)-2,$$

then the value of a is

(1) 3 (2) $-\frac{3}{2}$ (3) 2 (4) $\frac{3}{2}$

Ans. (1)

Sol. $I=\int_{-\pi/2}^{\pi/2}\left(\frac{x^2\cos x}{1+\pi^x}+\frac{1+\sin^2 x}{1+e^{\sin x^{2023}}}\right)dx$

$I=\int_{-\pi/2}^{\pi/2}\left(\frac{x^2\cos x}{1+\pi^{-x}}+\frac{1+\sin^2 x}{1+e^{\sin(-x)^{2023}}}\right)dx$

On Adding, we get

$2I=\int_{-\pi/2}^{\pi/2}\left(x^2\cos x+1+\sin^2 x\right)dx$

On solving

$I=\frac{\pi^2}{4}+\frac{3\pi}{4}-2$

a = 3

19. Suppose

$$f(x)=\frac{\left(2^x+2^{-x}\right)\tan x\sqrt{\tan^{-1}\left(x^2-x+1\right)}}{\left(7x^2+3x+1\right)^3},$$

Then the value of f '(0) is equal to

(1) π (2) 0

(3) $\sqrt{\pi}$ (4) $\frac{\pi}{2}$

Ans. (3)

Sol. $f'(0)=\lim_{h\to 0}\frac{f(h)-f(0)}{h}$

$=\lim_{h\to 0}\frac{(2^h+2^{-h})\tan h\sqrt{\tan^{-1}(h^2-h+1)}-0}{(7h^2+3h+1)^3 h}$

$=\sqrt{\pi}$

20. Let A be a square matrix such that $AA^T=I$. Then $\frac{1}{2}A\left[\left(A+A^T\right)^2+\left(A-A^T\right)^2\right]$ is equal to

(1) A^2+I (2) A^3+I

(3) A^2+A^T (4) A^3+A^T

Ans. (4)

Sol. $AA^T=I=A^TA$

On solving given expression, we get

$\frac{1}{2}A\left[A^2+(A^T)^2+2AA^T+A^2+(A^T)^2-2AA^T\right]$

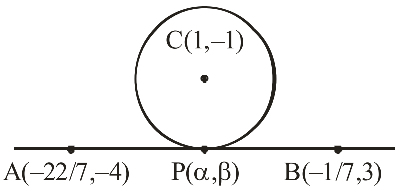
$=A[A^2+(A^T)^2]=A^3+A^T$

SECTION-B

21. Equation of two diameters of a circle are $2x-3y=5$ and $3x-4y=7$. The line joining the points $\left(-\frac{22}{7},-4\right)$ and $\left(-\frac{1}{7},3\right)$ intersects the circle at only one point $P(\alpha,\beta)$. Then $17\beta-\alpha$ is equal to

Ans. (2)

Sol. Centre of circle is (1, –1)



Equation of AB is 7x – 3y + 10 = 0 …(i)

Equation of CP is 3x + 7y + 4 = 0 …(ii)

Solving (i) and (ii)

$\alpha=\frac{-41}{29},\beta=\frac{1}{29}$ $\quad\therefore 17\beta-\alpha=2$

22. All the letters of the word "GTWENTY" are written in all possible ways with or without meaning and these words are written as in a dictionary. The serial number of the word "GTWENTY" IS

Ans. (553)

Sol. Words starting with E = 360

Words starting with GE = 60

Words starting with GN = 60

Words starting with GTE = 24

Words starting with GTN = 24

Words starting with GTT = 24

GTWENTY = 1

Total = 553

23. Let α, β be the roots of the equation $x^2 - x + 2 = 0$ with $\text{Im}(\alpha) > \text{Im}(\beta)$. Then $\alpha^6 + \alpha^4 + \beta^4 - 5\alpha^2$ is equal to

Ans. (13)

Sol. $\alpha^6 + \alpha^4 + \beta^4 - 5\alpha^2$

$= \alpha^4(\alpha - 2) + \alpha^4 - 5\alpha^2 + (\beta - 2)^2$

$= \alpha^5 - \alpha^4 - 5\alpha^2 + \beta^2 - 4\beta + 4$

$= \alpha^3(\alpha - 2) - \alpha^4 - 5\alpha^2 + \beta - 2 - 4\beta + 4$

$= -2\alpha^3 - 5\alpha^2 - 3\beta + 2$

$= -2\alpha(\alpha - 2) - 5\alpha^2 - 3\beta + 2$

$= -7\alpha^2 + 4\alpha - 3\beta + 2$

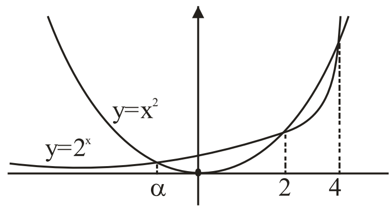
$= -7(\alpha - 2) + 4\alpha - 3\beta + 2$

$= -3\alpha - 3\beta + 16 = -3(1) + 16 = 13$

24. Let $f(x) = 2^x - x^2, x \in R$. If m and n are respectively the number of points at which the curves y = f(x) and y = f '(x) intersects the x-axis, then the value of m + n is

Ans. (5)

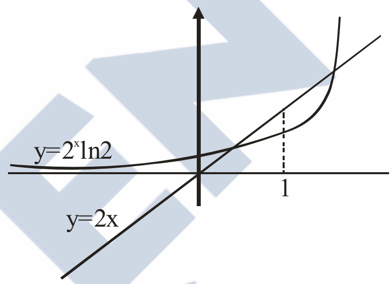
Sol.



$\therefore$ m = 3

$f'(x) = 2^x \ln 2 - 2x = 0$

$2^x \ln 2 = 2x$



$\therefore$ n = 2

$\Rightarrow$ m + n = 5

25. If the points of intersection of two distinct conics $x^2 + y^2 = 4b$ and $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ lie on the curve $y^2 = 3x^2$, then $3\sqrt{3}$ times the area of the rectangle formed by the intersection points is __

Ans. (432)

Sol. Putting $y^2 = 3x^2$ in both the conics

We get $x^2 = b$ and $\frac{b}{16} + \frac{3}{b} = 1$

$\Rightarrow b = 4, 12$ (b = 4 is rejected because curves coincide)

$\therefore$ b = 12

Hence points of intersection are

$\left(\pm\sqrt{12}, \pm 6\right) \Rightarrow$ area of rectangle = 432

26. If the solution curve $y = y(x)$ of the differential equation $(1+y^2)(1+\log_e x)dx + x\,dy = 0,\ x > 0$ passes through the point (1, 1) and $y(e) = \dfrac{\alpha - \tan\left(\frac{3}{2}\right)}{\beta + \tan\left(\frac{3}{2}\right)}$, then $\alpha + 2\beta$ is

Ans. (3)

Sol. $\int\left(\frac{1}{x}+\frac{\ln x}{x}\right)dx + \int\frac{dy}{1+y^2} = 0$

$$\ln x + \frac{(\ln x)^2}{2} + \tan^{-1} y = C$$

Put x = y = 1

$$\therefore C = \frac{\pi}{4}$$

$$\Rightarrow \ln x + \frac{(\ln x)^2}{2} + \tan^{-1} y = \frac{\pi}{4}$$

Put x = e

$$\Rightarrow y = \tan\left(\frac{\pi}{4} - \frac{3}{2}\right) = \frac{1 - \tan\frac{3}{2}}{1 + \tan\frac{3}{2}}$$

$$\therefore \alpha = 1, \beta = 1$$

$$\Rightarrow \alpha + 2\beta = 3$$

27. If the mean and variance of the data 65, 68, 58, 44, 48, 45, 60, $\alpha, \beta, 60$ where $\alpha > \beta$ are 56 and 66.2 respectively, then $\alpha^2 + \beta^2$ is equal to

Ans. (6344)

Sol. $\bar{x} = 56$

$$\sigma^2 = 66.2$$

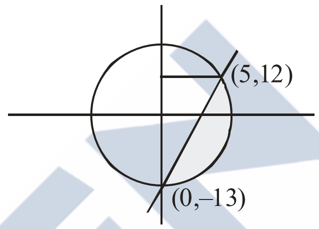
$$\Rightarrow \frac{\alpha^2 + \beta^2 + 25678}{10} - (56)^2 = 66.2$$

$$\therefore \alpha^2 + \beta^2 = 6344$$

28. The area (in sq. units) of the part of circle $x^2 + y^2 = 169$ which is below the line $5x - y = 13$ is $\frac{\pi\alpha}{2\beta} - \frac{65}{2} + \frac{\alpha}{\beta}\sin^{-1}\left(\frac{12}{13}\right)$ where α, β are coprime numbers. Then $\alpha + \beta$ is equal to

Ans. (171)

Sol.



$$\text{Area} = \int_{-13}^{12} \sqrt{169 - y^2}\,dy - \frac{1}{2}\times 25 \times 5$$

$$= \frac{\pi}{2}\times\frac{169}{2} - \frac{65}{2} + \frac{169}{2}\sin^{-1}\frac{12}{13}$$

$$\therefore \alpha + \beta = 171$$

29. If $\frac{{}^{11}C_1}{2} + \frac{{}^{11}C_2}{3} + \ldots\ldots + \frac{{}^{11}C_9}{10} = \frac{n}{m}$ with gcd(n, m) = 1, then n + m is equal to

Ans. (2041)

Sol. $\sum_{r=1}^{9} \frac{{}^{11}C_r}{r+1}$

$$= \frac{1}{12}\sum_{r=1}^{9} {}^{12}C_{r+1}$$

$$= \frac{1}{12}\left[2^{12} - 26\right] = \frac{2035}{6}$$

$$\therefore m + n = 2041$$

30. A line with direction ratios 2, 1, 2 meets the lines $x = y + 2 = z$ and $x + 2 = 2y = 2z$ respectively at the point P and Q. if the length of the perpendicular from the point (1, 2, 12) to the line PQ is l, then l^2 is

Ans. (65)

Sol. Let $P(t, t-2, t)$ and $Q(2s-2, s, s)$

D.R's of PQ are 2, 1, 2

$$\frac{2s-2-t}{2} = \frac{s-t+2}{1} = \frac{s-t}{2}$$

$\Rightarrow t = 6$ and $s = 2$

$\Rightarrow P(6,4,6)$ and $Q(2,2,2)$

$$PQ : \frac{x-2}{2} = \frac{y-2}{1} = \frac{z-2}{2} = \lambda$$

Let $F(2\lambda+2, \lambda+2, 2\lambda+2)$

$A(1,2,12)$

$\overrightarrow{AF} \cdot \overrightarrow{PQ} = 0$

$\therefore \lambda = 2$

So F(6,4, 6) and $AF = \sqrt{65}$