

FINAL JEE-MAIN EXAMINATION – JANUARY, 2024

(Held On Saturday 27th January, 2024)

TIME : 3 : 00 PM to 6 : 00 PM

MATHEMATICS

TEST PAPER WITH SOLUTION

SECTION-A

1. Considering only the principal values of inverse trigonometric functions, the number of positive real values of x satisfying $\tan^{-1}(x) + \tan^{-1}(2x) = \frac{\pi}{4}$

is :

- (1) More than 2
- (2) 1
- (3) 2
- (4) 0

Ans. (2)

Sol. $\tan^{-1} x + \tan^{-1} 2x = \frac{\pi}{4}; x > 0$

$\Rightarrow \tan^{-1} 2x = \frac{\pi}{4} - \tan^{-1} x$

Taking tan both sides

$\Rightarrow 2x = \frac{1-x}{1+x}$

$\Rightarrow 2x^2 + 3x - 1 = 0$

$x = \frac{-3 \pm \sqrt{9+8}}{8} = \frac{-3 \pm \sqrt{17}}{8}$

Only possible $x = \frac{-3 + \sqrt{17}}{8}$

2. Consider the function $f : (0,2) \rightarrow \mathbb{R}$ defined by

$f(x) = \frac{x}{2} + \frac{2}{x}$ and the function $g(x)$ defined by

$g(x) = \begin{cases} \min\{f(t)\}, & 0 < t \leq x \text{ and } 0 < x \leq 1 \\ \frac{3}{2} + x, & 1 < x < 2 \end{cases}$. Then

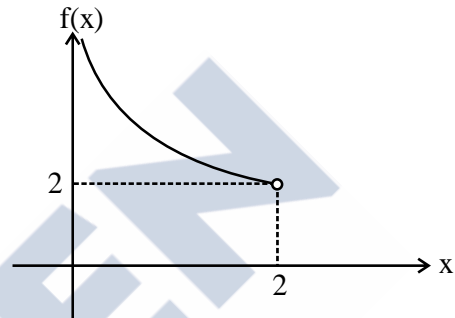
- (1) g is continuous but not differentiable at $x = 1$
- (2) g is not continuous for all $x \in (0,2)$
- (3) g is neither continuous nor differentiable at $x = 1$
- (4) g is continuous and differentiable for all $x \in (0,2)$

Ans. (1)

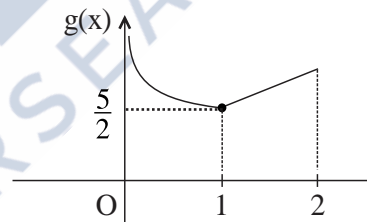
Sol. $f : (0,2) \rightarrow \mathbb{R}; f(x) = \frac{x}{2} + \frac{2}{x}$

$f'(x) = \frac{1}{2} - \frac{2}{x^2}$

$\therefore f(x)$ is decreasing in domain.



$g(x) = \begin{cases} \frac{x}{2} + \frac{2}{x} & 0 < x \leq 1 \\ \frac{3}{2} + x & 1 < x < 2 \end{cases}$



3. Let the image of the point $(1, 0, 7)$ in the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ be the point (α, β, γ) . Then

which one of the following points lies on the line passing through (α, β, γ) and making angles $\frac{2\pi}{3}$

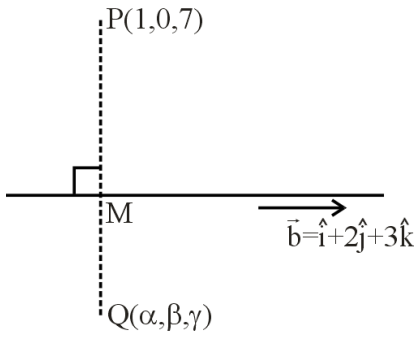
and $\frac{3\pi}{4}$ with y -axis and z -axis respectively and an acute angle with x -axis ?

- (1) $(1, -2, 1 + \sqrt{2})$
- (2) $(1, 2, 1 - \sqrt{2})$
- (3) $(3, 4, 3 - 2\sqrt{2})$
- (4) $(3, -4, 3 + 2\sqrt{2})$

Ans. (3)



Sol. $L_1 = \frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3} = \lambda$



$M(\lambda, 1+2\lambda, 2+3\lambda)$

$\vec{PM} = (\lambda - 1)\hat{i} + (1 + 2\lambda)\hat{j} + (3\lambda - 5)\hat{k}$

\vec{PM} is perpendicular to line L_1

$\vec{PM} \cdot \vec{b} = 0 \quad (\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k})$

$\Rightarrow \lambda - 1 + 4\lambda + 2 + 9\lambda - 15 = 0$

$14\lambda = 14 \Rightarrow \lambda = 1$

$\therefore M = (1, 3, 5)$

$\vec{Q} = 2\vec{M} - \vec{P}$ [M is midpoint of \vec{P} & \vec{Q}]

$\vec{Q} = 2\hat{i} + 6\hat{j} + 10\hat{k} - \hat{i} - 7\hat{k}$

$\vec{Q} = \hat{i} + 6\hat{j} + 3\hat{k}$

$\therefore (\alpha, \beta, \gamma) = (1, 6, 3)$

Required line having direction cosine (l, m, n)

$l^2 + m^2 + n^2 = 1$

$\Rightarrow l^2 + \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2 = 1$

$l^2 = \frac{1}{4}$

$\therefore l = \frac{1}{2}$ [Line make acute angle with x-axis]

Equation of line passing through $(1, 6, 3)$ will be

$\vec{r} = (\hat{i} + 6\hat{j} + 3\hat{k}) + \mu \left(\frac{1}{2}\hat{i} - \frac{1}{2}\hat{j} - \frac{1}{\sqrt{2}}\hat{k} \right)$

Option (3) satisfying for $\mu = 4$

4. Let R be the interior region between the lines $3x - y + 1 = 0$ and $x + 2y - 5 = 0$ containing the origin. The set of all values of a, for which the points $(a^2, a + 1)$ lie in R, is :

(1) $(-3, -1) \cup \left(-\frac{1}{3}, 1\right)$

(2) $(-3, 0) \cup \left(\frac{1}{3}, 1\right)$

(3) $(-3, 0) \cup \left(\frac{2}{3}, 1\right)$

(4) $(-3, -1) \cup \left(\frac{1}{3}, 1\right)$

Ans. (2)

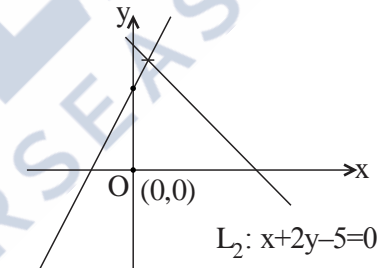
Sol. $P(a^2, a + 1)$

$L_1 = 3x - y + 1 = 0$

Origin and P lies same side w.r.t. L_1

$\Rightarrow L_1(0) \cdot L_1(P) > 0$

$\therefore 3(a^2) - (a + 1) + 1 > 0$



$L_1: 3x - y + 1 = 0$

$\Rightarrow 3a^2 - a > 0$

$a \in (-\infty, 0) \cup \left(\frac{1}{3}, \infty\right)$ (1)

Let $L_2: x + 2y - 5 = 0$

Origin and P lies same side w.r.t. L_2

$\Rightarrow L_2(0) \cdot L_2(P) > 0$

$\Rightarrow a^2 + 2(a + 1) - 5 < 0$

$\Rightarrow a^2 + 2a - 3 < 0$

$\Rightarrow (a + 3)(a - 1) < 0$

$\therefore a \in (-3, 1)$ (2)

Intersection of (1) and (2)

$a \in (-3, 0) \cup \left(\frac{1}{3}, 1\right)$



5. The 20th term from the end of the progression

$$20, 19\frac{1}{4}, 18\frac{1}{2}, 17\frac{3}{4}, \dots, -129\frac{1}{4} \text{ is :-}$$

- (1) -118
- (2) -110
- (3) -115
- (4) -100

Ans. (3)

Sol. $20, 19\frac{1}{4}, 18\frac{1}{2}, 17\frac{3}{4}, \dots, -129\frac{1}{4}$

This is A.P. with common difference

$$d_1 = -1 + \frac{1}{4} = -\frac{3}{4}$$

$$-129\frac{1}{4}, \dots, 19\frac{1}{4}, 20$$

This is also A.P. $a = -129\frac{1}{4}$ and $d = \frac{3}{4}$

Required term =

$$-129\frac{1}{4} + (20-1)\left(\frac{3}{4}\right)$$

$$= -129 - \frac{1}{4} + 15 - \frac{3}{4} = -115$$

6. Let $f: \mathbb{R} - \left\{-\frac{1}{2}\right\} \rightarrow \mathbb{R}$ and $g: \mathbb{R} - \left\{-\frac{5}{2}\right\} \rightarrow \mathbb{R}$ be

defined as $f(x) = \frac{2x+3}{2x+1}$ and $g(x) = \frac{|x|+1}{2x+5}$. Then

the domain of the function fog is :

- (1) $\mathbb{R} - \left\{-\frac{5}{2}\right\}$
- (2) \mathbb{R}
- (3) $\mathbb{R} - \left\{-\frac{7}{4}\right\}$
- (4) $\mathbb{R} - \left\{-\frac{5}{2}, -\frac{7}{4}\right\}$

Ans. (1)

Sol. $f(x) = \frac{2x+3}{2x+1}; x \neq -\frac{1}{2}$

$$g(x) = \frac{|x|+1}{2x+5}, x \neq -\frac{5}{2}$$

Domain of f(g(x))

$$f(g(x)) = \frac{2g(x)+3}{2g(x)+1}$$

$$x \neq -\frac{5}{2} \text{ and } \frac{|x|+1}{2x+5} \neq -\frac{1}{2}$$

$$x \in \mathbb{R} - \left\{-\frac{5}{2}\right\} \text{ and } x \in \mathbb{R}$$

$$\therefore \text{Domain will be } \mathbb{R} - \left\{-\frac{5}{2}\right\}$$

7. For $0 < a < 1$, the value of the integral

$$\int_0^\pi \frac{dx}{1-2a \cos x + a^2} \text{ is :}$$

(1) $\frac{\pi^2}{\pi+a^2}$

(2) $\frac{\pi^2}{\pi-a^2}$

(3) $\frac{\pi}{1-a^2}$

(4) $\frac{\pi}{1+a^2}$

Ans. (3)

Sol. $I = \int_0^\pi \frac{dx}{1-2a \cos x + a^2}; 0 < a < 1$

$$I = \int_0^\pi \frac{dx}{1+2a \cos x + a^2}$$

$$2I = 2 \int_0^{\pi/2} \frac{2(1+a^2)}{(1+a^2)^2 - 4a^2 \cos^2 x} dx$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{2(1+a^2) \cdot \sec^2 x}{(1+a^2)^2 \cdot \sec^2 x - 4a^2} dx$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{2 \cdot (1+a^2) \cdot \sec^2 x}{(1+a^2)^2 \cdot \tan^2 x + (1-a^2)^2} dx$$



$$\Rightarrow I = \int_0^{\pi/2} \frac{2 \cdot \sec^2 x}{\tan^2 x + \left(\frac{1-a^2}{1+a^2}\right)^2} \cdot dx$$

$$\Rightarrow I = \frac{2}{(1-a^2)} \left[\frac{\pi}{2} - 0 \right]$$

$$I = \frac{\pi}{1-a^2}$$

8. Let $g(x) = 3f\left(\frac{x}{3}\right) + f(3-x)$ and $f''(x) > 0$ for all $x \in (0, 3)$. If g is decreasing in $(0, \alpha)$ and increasing in $(\alpha, 3)$, then 8α is

- (1) 24
(2) 0
(3) 18
(4) 20

Ans. (3)

Sol. $g(x) = 3f\left(\frac{x}{3}\right) + f(3-x)$ and $f''(x) > 0 \forall x \in (0, 3)$

$\Rightarrow f'(x)$ is increasing function

$$g'(x) = 3 \times \frac{1}{3} \cdot f'\left(\frac{x}{3}\right) - f'(3-x)$$

$$= f'\left(\frac{x}{3}\right) - f'(3-x)$$

If g is decreasing in $(0, \alpha)$

$$g'(x) < 0$$

$$f'\left(\frac{x}{3}\right) - f'(3-x) < 0$$

$$f'\left(\frac{x}{3}\right) < f'(3-x)$$

$$\Rightarrow \frac{x}{3} < 3-x$$

$$\Rightarrow x < \frac{9}{4}$$

$$\text{Therefore } \alpha = \frac{9}{4}$$

$$\text{Then } 8\alpha = 8 \times \frac{9}{4} = 18$$

9. If $\lim_{x \rightarrow 0} \frac{3 + \alpha \sin x + \beta \cos x + \log_e(1-x)}{3 \tan^2 x} = \frac{1}{3}$, then

$2\alpha - \beta$ is equal to :

- (1) 2
(2) 7
(3) 5
(4) 1

Ans. (3)

Sol. $\lim_{x \rightarrow 0} \frac{3 + \alpha \sin x + \beta \cos x + \log_e(1-x)}{3 \tan^2 x} = \frac{1}{3}$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{3 + \alpha \left[x - \frac{x^3}{3!} + \dots \right] + \beta \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \right] + \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \dots \right)}{3 \tan^2 x} = \frac{1}{3}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{(3 + \beta) + (\alpha - 1)x + \left(-\frac{1}{2} - \frac{\beta}{2} \right) x^2 + \dots}{3x^2} \times \frac{x^2}{\tan^2 x} = \frac{1}{3}$$

$$\Rightarrow \beta + 3 = 0, \alpha - 1 = 0 \text{ and } \frac{-\frac{1}{2} - \frac{\beta}{2}}{3} = \frac{1}{3}$$

$$\Rightarrow \beta = -3, \alpha = 1$$

$$\Rightarrow 2\alpha - \beta = 2 + 3 = 5$$

10. If α, β are the roots of the equation, $x^2 - x - 1 = 0$ and $S_n = 2023\alpha^n + 2024\beta^n$, then

- (1) $2S_{12} = S_{11} + S_{10}$
(2) $S_{12} = S_{11} + S_{10}$
(3) $2S_{11} = S_{12} + S_{10}$
(4) $S_{11} = S_{10} + S_{12}$

Ans. (2)

Sol. $x^2 - x - 1 = 0$

$$S_n = 2023\alpha^n + 2024\beta^n$$

$$S_{n-1} + S_{n-2} = 2023\alpha^{n-1} + 2024\beta^{n-1} + 2023\alpha^{n-2} + 2024\beta^{n-2}$$

$$= 2023\alpha^{n-2}[1 + \alpha] + 2024\beta^{n-2}[1 + \beta]$$

$$= 2023\alpha^{n-2}[\alpha^2] + 2024\beta^{n-2}[\beta^2]$$

$$= 2023\alpha^n + 2024\beta^n$$

$$S_{n-1} + S_{n-2} = S_n$$

Put $n = 12$

$$S_{11} + S_{10} = S_{12}$$



11. Let A and B be two finite sets with m and n elements respectively. The total number of subsets of the set A is 56 more than the total number of subsets of B. Then the distance of the point P(m, n) from the point Q(-2, -3) is

- (1) 10
- (2) 6
- (3) 4
- (4) 8

Ans. (1)

Sol. $2^m - 2^n = 56$

$$2^n(2^{m-n} - 1) = 2^3 \times 7$$

$$2^n = 2^3 \text{ and } 2^{m-n} - 1 = 7$$

$$\Rightarrow n = 3 \text{ and } 2^{m-n} = 8$$

$$\Rightarrow n = 3 \text{ and } m - n = 3$$

$$\Rightarrow n = 3 \text{ and } m = 6$$

$$P(6,3) \text{ and } Q(-2, -3)$$

$$PQ = \sqrt{8^2 + 6^2} = \sqrt{100} = 10$$

Hence option (1) is correct

12. The values of α , for which

$$\begin{vmatrix} 1 & \frac{3}{2} & \alpha + \frac{3}{2} \\ 1 & \frac{1}{3} & \alpha + \frac{1}{3} \\ 2\alpha + 3 & 3\alpha + 1 & 0 \end{vmatrix} = 0, \text{ lie in the interval}$$

- (1) (-2, 1)
- (2) (-3, 0)
- (3) $\left(-\frac{3}{2}, \frac{3}{2}\right)$
- (4) (0, 3)

Ans. (2)

Sol.

$$\begin{vmatrix} 1 & \frac{3}{2} & \alpha + \frac{3}{2} \\ 1 & \frac{1}{3} & \alpha + \frac{1}{3} \\ 2\alpha + 3 & 3\alpha + 1 & 0 \end{vmatrix} = 0$$

$$\Rightarrow (2\alpha + 3) \left\{ \frac{7\alpha}{6} \right\} - (3\alpha + 1) \left\{ \frac{-7}{6} \right\} = 0$$

$$\Rightarrow (2\alpha + 3) \cdot \frac{7\alpha}{6} + (3\alpha + 1) \cdot \frac{7}{6} = 0$$

$$\Rightarrow 2\alpha^2 + 3\alpha + 3\alpha + 1 = 0$$

$$\Rightarrow 2\alpha^2 + 6\alpha + 1 = 0$$

$$\Rightarrow \alpha = \frac{-3 + \sqrt{7}}{2}, \frac{-3 - \sqrt{7}}{2}$$

Hence option (2) is correct.

13. An urn contains 6 white and 9 black balls. Two successive draws of 4 balls are made without replacement. The probability, that the first draw gives all white balls and the second draw gives all black balls, is :

- (1) $\frac{5}{256}$
- (2) $\frac{5}{715}$
- (3) $\frac{3}{715}$
- (4) $\frac{3}{256}$

Ans. (3)

Sol. $\frac{{}^6C_4}{{}^{15}C_4} \times \frac{{}^9C_4}{{}^{11}C_4} = \frac{3}{715}$

Hence option (3) is correct.

14. The integral $\int \frac{(x^8 - x^2)dx}{(x^{12} + 3x^6 + 1) \tan^{-1}\left(x^3 + \frac{1}{x^3}\right)}$ is

equal to :

- (1) $\log_e \left(\left| \tan^{-1}\left(x^3 + \frac{1}{x^3}\right) \right| \right)^{1/3} + C$
- (2) $\log_e \left(\left| \tan^{-1}\left(x^3 + \frac{1}{x^3}\right) \right| \right)^{1/2} + C$
- (3) $\log_e \left(\left| \tan^{-1}\left(x^3 + \frac{1}{x^3}\right) \right| \right) + C$
- (4) $\log_e \left(\left| \tan^{-1}\left(x^3 + \frac{1}{x^3}\right) \right| \right)^3 + C$

Ans. (1)



Sol. $I = \int \frac{x^8 - x^2}{(x^{12} + 3x^6 + 1) \tan^{-1}\left(x^3 + \frac{1}{x^3}\right)} dx$

Let $\tan^{-1}\left(x^3 + \frac{1}{x^3}\right) = t$

$\Rightarrow \frac{1}{1 + \left(x^3 + \frac{1}{x^3}\right)^2} \cdot \left(3x^2 - \frac{3}{x^4}\right) dx = dt$

$\Rightarrow \frac{x^6}{x^{12} + 3x^6 + 1} \cdot \frac{3x^6 - 3}{x^4} dx = dt$

$I = \frac{1}{3} \int \frac{dt}{t} = \frac{1}{3} \ln |t| + C$

$I = \frac{1}{3} \ln \left| \tan^{-1}\left(x^3 + \frac{1}{x^3}\right) \right| + C$

$I = \ln \left| \tan^{-1}\left(x^3 + \frac{1}{x^3}\right) \right|^{1/3} + C$

Hence option (1) is correct

15. If $2 \tan^2 \theta - 5 \sec \theta = 1$ has exactly 7 solutions in the interval $\left[0, \frac{n\pi}{2}\right]$, for the least value of $n \in \mathbb{N}$

then $\sum_{k=1}^n \frac{k}{2^k}$ is equal to :

(1) $\frac{1}{2^{15}}(2^{14} - 14)$

(2) $\frac{1}{2^{14}}(2^{15} - 15)$

(3) $1 - \frac{15}{2^{13}}$

(4) $\frac{1}{2^{13}}(2^{14} - 15)$

Ans. (4)

Sol. $2 \tan^2 \theta - 5 \sec \theta - 1 = 0$
 $\Rightarrow 2 \sec^2 \theta - 5 \sec \theta - 3 = 0$
 $\Rightarrow (2 \sec \theta + 1)(\sec \theta - 3) = 0$
 $\Rightarrow \sec \theta = -\frac{1}{2}, 3$
 $\Rightarrow \cos \theta = -2, \frac{1}{3}$
 $\Rightarrow \cos \theta = \frac{1}{3}$

For 7 solutions $n = 13$

So, $\sum_{k=1}^{13} \frac{k}{2^k} = S$ (say)

$S = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \dots + \frac{13}{2^{13}}$

$\frac{1}{2} S = \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{12}{2^{13}} + \frac{13}{2^{14}}$

$\Rightarrow \frac{S}{2} = \frac{1}{2} \cdot \frac{1 - \frac{1}{2^{13}}}{1 - \frac{1}{2}} - \frac{13}{2^{14}} \Rightarrow S = 2 \cdot \left(\frac{2^{13} - 1}{2^{13}} \right) - \frac{13}{2^{13}}$

16. The position vectors of the vertices A, B and C of a triangle are $2\hat{i} - 3\hat{j} + 3\hat{k}$, $2\hat{i} + 2\hat{j} + 3\hat{k}$ and $-\hat{i} + \hat{j} + 3\hat{k}$ respectively. Let l denotes the length of the angle bisector AD of $\angle BAC$ where D is on the line segment BC, then $2l^2$ equals :

(1) 49

(2) 42

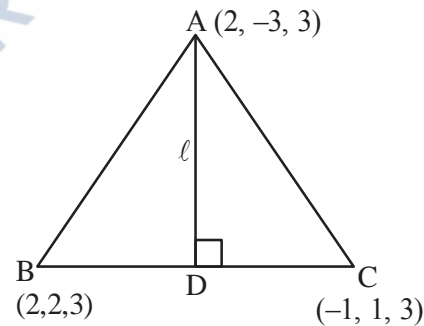
(3) 50

(4) 45

Ans. (4)

Sol. $AB = 5$

$AC = 5$



\therefore D is midpoint of BC

$D\left(\frac{1}{2}, \frac{3}{2}, 3\right)$

$\therefore l = \sqrt{\left(2 - \frac{1}{2}\right)^2 + \left(-3 - \frac{3}{2}\right)^2 + (3 - 3)^2}$

$l = \sqrt{\frac{45}{2}}$

$\therefore 2l^2 = 45$



17. If $y = y(x)$ is the solution curve of the differential equation $(x^2 - 4)dy - (y^2 - 3y)dx = 0$, $x > 2, y(4) = \frac{3}{2}$ and the slope of the curve is never zero, then the value of $y(10)$ equals :

- (1) $\frac{3}{1+(8)^{1/4}}$
- (2) $\frac{3}{1+2\sqrt{2}}$
- (3) $\frac{3}{1-2\sqrt{2}}$
- (4) $\frac{3}{1-(8)^{1/4}}$

Ans. (1)

Sol. $(x^2 - 4)dy - (y^2 - 3y)dx = 0$

$$\Rightarrow \int \frac{dy}{y^2 - 3y} = \int \frac{dx}{x^2 - 4}$$

$$\Rightarrow \frac{1}{3} \int \frac{y - (y - 3)}{y(y - 3)} dy = \int \frac{dx}{x^2 - 4}$$

$$\Rightarrow \frac{1}{3} (\ln |y - 3| - \ln |y|) = \frac{1}{4} \ln \left| \frac{x - 2}{x + 2} \right| + C$$

$$\Rightarrow \frac{1}{3} \ln \left| \frac{y - 3}{y} \right| = \frac{1}{4} \ln \left| \frac{x - 2}{x + 2} \right| + C$$

At $x = 4, y = \frac{3}{2}$

$$\therefore C = \frac{1}{4} \ln 3$$

$$\therefore \frac{1}{3} \ln \left| \frac{y - 3}{y} \right| = \frac{1}{4} \ln \left| \frac{x - 2}{x + 2} \right| + \frac{1}{4} \ln 3$$

At $x = 10$

$$\frac{1}{3} \ln \left| \frac{y - 3}{y} \right| = \frac{1}{4} \ln \left| \frac{2}{3} \right| + \frac{1}{4} \ln 3$$

$$\ln \left| \frac{y - 3}{y} \right| = \ln 2^{3/4}, \forall x > 2, \frac{dy}{dx} < 0$$

as $y(4) = \frac{3}{2} \Rightarrow y \in (0, 3)$

$$-y + 3 = 8^{1/4} \cdot y$$

$$y = \frac{3}{1 + 8^{1/4}}$$

18. Let e_1 be the eccentricity of the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ and e_2 be the eccentricity of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$, which passes through the foci of the hyperbola. If $e_1 e_2 = 1$, then the length of the chord of the ellipse parallel to the x-axis and passing through $(0, 2)$ is :

- (1) $4\sqrt{5}$
- (2) $\frac{8\sqrt{5}}{3}$
- (3) $\frac{10\sqrt{5}}{3}$
- (4) $3\sqrt{5}$

Ans. (3)

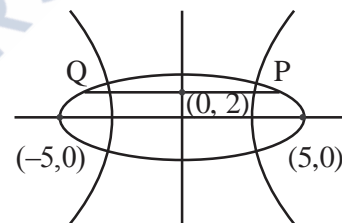
Sol. H: $\frac{x^2}{16} - \frac{y^2}{9} = 1$ $e_1 = \frac{5}{4}$

$$\therefore e_1 e_2 = 1 \Rightarrow e_2 = \frac{4}{5}$$

Also, ellipse is passing through $(\pm 5, 0)$

$$\therefore a = 5 \text{ and } b = 3$$

$$E: \frac{x^2}{25} + \frac{y^2}{9} = 1$$



End point of chord are $\left(\pm \frac{5\sqrt{5}}{3}, 2 \right)$

$$\therefore L_{PQ} = \frac{10\sqrt{5}}{3}$$

19. Let $\alpha = \frac{(4!)!}{(4!)^{3!}}$ and $\beta = \frac{(5!)!}{(5!)^{4!}}$. Then :

- (1) $\alpha \in \mathbb{N}$ and $\beta \notin \mathbb{N}$
- (2) $\alpha \notin \mathbb{N}$ and $\beta \in \mathbb{N}$
- (3) $\alpha \in \mathbb{N}$ and $\beta \in \mathbb{N}$
- (4) $\alpha \notin \mathbb{N}$ and $\beta \notin \mathbb{N}$

Ans. (3)



Sol. $\alpha = \frac{(4!)!}{(4!)^{3!}}, \beta = \frac{(5!)!}{(5!)^{4!}}$
 $\alpha = \frac{(24)!}{(4!)^6}, \beta = \frac{(120)!}{(5!)^{24}}$

Let 24 distinct objects are divided into 6 groups of 4 objects in each group.

No. of ways of formation of group = $\frac{24!}{(4!)^6 \cdot 6!} \in \mathbb{N}$

Similarly,

Let 120 distinct objects are divided into 24 groups of 5 objects in each group.

No. of ways of formation of groups

= $\frac{(120)!}{(5!)^{24} \cdot 24!} \in \mathbb{N}$

20. Let the position vectors of the vertices A, B and C of a triangle be $2\hat{i} + 2\hat{j} + \hat{k}$, $\hat{i} + 2\hat{j} + 2\hat{k}$ and $2\hat{i} + \hat{j} + 2\hat{k}$ respectively. Let l_1, l_2 and l_3 be the lengths of perpendiculars drawn from the orthocenter of the triangle on the sides AB, BC and CA respectively, then $l_1^2 + l_2^2 + l_3^2$ equals :

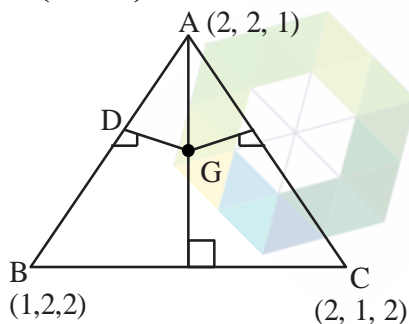
- (1) $\frac{1}{5}$ (2) $\frac{1}{2}$
 (3) $\frac{1}{4}$ (4) $\frac{1}{3}$

Ans. (2)

Sol. ΔABC is equilateral

Orthocentre and centroid will be same

$G\left(\frac{5}{3}, \frac{5}{3}, \frac{5}{3}\right)$



Mid-point of AB is $D\left(\frac{3}{2}, 2, \frac{3}{2}\right)$

$\therefore l_1 = \sqrt{\frac{1}{36} + \frac{1}{9} + \frac{1}{36}}$

$l_1 = \sqrt{\frac{1}{6}} = l_2 = l_3$

$\therefore l_1^2 + l_2^2 + l_3^2 = \frac{1}{2}$

SECTION-B

21. The mean and standard deviation of 15 observations were found to be 12 and 3 respectively. On rechecking it was found that an observation was read as 10 in place of 12. If μ and σ^2 denote the mean and variance of the correct observations respectively, then $15(\mu + \mu^2 + \sigma^2)$ is equal to

Ans. (2521)

Sol. Let the incorrect mean be μ' and standard deviation be σ'

We have

$\mu' = \frac{\sum x_i}{15} = 12 \Rightarrow \sum x_i = 180$

As per given information correct $\sum x_i = 180 - 10 + 12$

$\Rightarrow \mu$ (correct mean) = $\frac{182}{15}$

Also

$\sigma' = \sqrt{\frac{\sum x_i^2}{15} - 144} = 3 \Rightarrow \sum x_i^2 = 2295$

Correct $\sum x_i^2 = 2295 - 100 + 144 = 2339$

σ^2 (correct variance) = $\frac{2339}{15} - \frac{182 \times 182}{15 \times 15}$

Required value

= $15(\mu + \mu^2 + \sigma^2)$

= $15\left(\frac{182}{15} + \frac{182 \times 182}{15 \times 15} + \frac{2339}{15} - \frac{182 \times 182}{15 \times 15}\right)$

= $15\left(\frac{182}{15} + \frac{2339}{15}\right)$

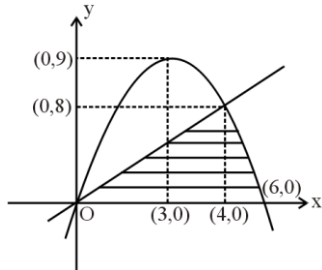
= 2521



22. If the area of the region $\{(x, y) : 0 \leq y \leq \min\{2x, 6x - x^2\}\}$ is A, then $12A$ is equal to.....

Ans. (304)

Sol. We have



$$A = \frac{1}{2} \times 4 \times 8 + \int_4^6 (6x - x^2) dx$$

$$A = \frac{76}{3}$$

$$12A = 304$$

23. Let A be a 2×2 real matrix and I be the identity matrix of order 2. If the roots of the equation $|A - xI| = 0$ be -1 and 3 , then the sum of the diagonal elements of the matrix A^2 is.....

Ans. (10)

Sol. $|A - xI| = 0$

Roots are -1 and 3

Sum of roots = $\text{tr}(A) = 2$

Product of roots = $|A| = -3$

$$\text{Let } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

We have $a + d = 2$

$ad - bc = -3$

$$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix}$$

We need $a^2 + bc + bc + d^2$

$$= a^2 + 2bc + d^2$$

$$= (a + d)^2 - 2ad + 2bc$$

$$= 4 - 2(ad - bc)$$

$$= 4 - 2(-3)$$

$$= 4 + 6$$

$$= 10$$

24. If the sum of squares of all real values of α , for which the lines $2x - y + 3 = 0$, $6x + 3y + 1 = 0$ and $\alpha x + 2y - 2 = 0$ do not form a triangle is p, then the greatest integer less than or equal to p is

Ans. (32)

Sol. $2x - y + 3 = 0$

$$6x + 3y + 1 = 0$$

$$\alpha x + 2y - 2 = 0$$

Will not form a Δ if $\alpha x + 2y - 2 = 0$ is concurrent with $2x - y + 3 = 0$ and $6x + 3y + 1 = 0$ or parallel to either of them so

Case-1: Concurrent lines

$$\begin{vmatrix} 2 & -1 & 3 \\ 6 & 3 & 1 \\ \alpha & 2 & -2 \end{vmatrix} = 0 \Rightarrow \alpha = \frac{4}{5}$$

Case-2 : Parallel lines

$$-\frac{\alpha}{2} = \frac{-6}{3} \text{ or } -\frac{\alpha}{2} = 2$$

$$\Rightarrow \alpha = 4 \text{ or } \alpha = -4$$

$$P = 16 + 16 + \frac{16}{25}$$

$$[P] = \left[32 + \frac{16}{25} \right] = 32$$

25. The coefficient of x^{2012} in the expansion of $(1-x)^{2008}(1+x+x^2)^{2007}$ is equal to

Ans. (0)



Sol. $(1-x)(1-x)^{2007}(1+x+x^2)^{2007}$

$(1-x)(1-x^3)^{2007}$

$(1-x)({}^{2007}C_0 - {}^{2007}C_1(x^3) + \dots)$

General term

$(1-x)((-1)^r {}^{2007}C_r x^{3r})$

$(-1)^r {}^{2007}C_r x^{3r} - (-1)^r {}^{2007}C_r x^{3r+1}$

$3r = 2012$

$r \neq \frac{2012}{3}$

$3r + 1 = 2012$

$3r = 2011$

$r \neq \frac{2011}{3}$

Hence there is no term containing x^{2012} .

So coefficient of $x^{2012} = 0$

26. If the solution curve, of the differential equation

$\frac{dy}{dx} = \frac{x+y-2}{x-y}$ passing through the point (2, 1) is

$\tan^{-1}\left(\frac{y-1}{x-1}\right) - \frac{1}{\beta} \log_e \left(\alpha + \left(\frac{y-1}{x-1}\right)^2 \right) = \log_e |x-1|,$

then $5\beta + \alpha$ is equal to

Ans. (11)

Sol. $\frac{dy}{dx} = \frac{x+y-2}{x-y}$

$x = X + h, y = Y + k$

$\frac{dY}{dX} = \frac{X+Y}{X-Y}$

$\left. \begin{matrix} h+k-2=0 \\ h-k=0 \end{matrix} \right\} h=k=1$

$Y = vX$

$v + \frac{dv}{dX} = \frac{1+v}{1-v} \Rightarrow X - \frac{dv}{dX} = \frac{1+v^2}{1-v}$

$\frac{1-v}{1+v^2} dv = \frac{dX}{X}$

$\tan^{-1} v - \frac{1}{2} \ln(1+v^2) = \ln |X| + C$

As curve is passing through (2, 1)

$\tan^{-1}\left(\frac{y-1}{x-1}\right) - \frac{1}{2} \ln\left(1 + \left(\frac{y-1}{x-1}\right)^2\right) = \ln |x-1|$

$\therefore \alpha = 1$ and $\beta = 2$

$\Rightarrow 5\beta + \alpha = 11$

27. Let $f(x) = \int_0^x g(t) \log_e \left(\frac{1-t}{1+t} \right) dt$, where g is a continuous odd function.

If $\int_{-\pi/2}^{\pi/2} \left(f(x) + \frac{x^2 \cos x}{1+e^x} \right) dx = \left(\frac{\pi}{\alpha} \right)^2 - \alpha$, then α is equal to.....

Ans. (2)

Sol. $f(x) = \int_0^x g(t) \ln \left(\frac{1-t}{1+t} \right) dt$

$f(-x) = \int_0^{-x} g(t) \ln \left(\frac{1-t}{1+t} \right) dt$

$f(-x) = -\int_0^x g(-y) \ln \left(\frac{1+y}{1-y} \right) dy$

$= -\int_0^x g(y) \ln \left(\frac{1-y}{1+y} \right) dy$ (g is odd)

$f(-x) = -f(x) \Rightarrow f$ is also odd

Now,

$I = \int_{-\pi/2}^{\pi/2} \left(f(x) + \frac{x^2 \cos x}{1+e^x} \right) dx \dots (1)$

$I = \int_{-\pi/2}^{\pi/2} \left(f(-x) + \frac{x^2 e^x \cos x}{1+e^x} \right) dx \dots (2)$

$2I = \int_{-\pi/2}^{\pi/2} x^2 \cos x dx = 2 \int_0^{\pi/2} x^2 \cos x dx$



$$I = (x^2 \sin x)_0^{\pi/2} - \int_0^{\pi/2} 2x \sin x dx$$

$$= \frac{\pi^2}{4} - 2(-x \cos x + \int \cos x dx)_0^{\pi/2}$$

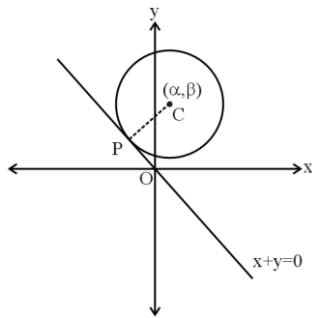
$$= \frac{\pi^2}{4} - 2(0+1) = \frac{\pi^2}{4} - 2 \Rightarrow \left(\frac{\pi}{2}\right)^2 - 2$$

$\therefore \alpha = 2$

28. Consider a circle $(x-\alpha)^2 + (y-\beta)^2 = 50$, where $\alpha, \beta > 0$. If the circle touches the line $y + x = 0$ at the point P, whose distance from the origin is $4\sqrt{2}$, then $(\alpha + \beta)^2$ is equal to.....

Ans. (100)

Sol.



S: $(x-\alpha)^2 + (y-\beta)^2 = 50$

CP = r

$$\left| \frac{\alpha + \beta}{\sqrt{2}} \right| = 5\sqrt{2}$$

$$\Rightarrow (\alpha + \beta)^2 = 100$$

29. The lines $\frac{x-2}{2} = \frac{y}{-2} = \frac{z-7}{16}$ and

$$\frac{x+3}{4} = \frac{y+2}{3} = \frac{z+2}{1}$$

intersect at the point P. If the

distance of P from the line $\frac{x+1}{2} = \frac{y-1}{3} = \frac{z-1}{1}$ is l ,

then $14l^2$ is equal to.....

Ans. (108)

Sol. $\frac{x-2}{1} = \frac{y}{-1} = \frac{z-7}{8} = \lambda$

$$\frac{x+3}{4} = \frac{y+2}{3} = \frac{z+2}{1} = k$$

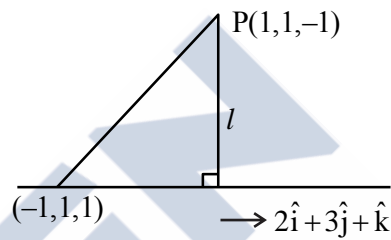
$$\Rightarrow \lambda + 2 = 4k - 3$$

$$-\lambda = 3k - 2$$

$$\Rightarrow k = 1, \lambda = -1$$

$$8\lambda + 7 = k - 2$$

$\therefore P = (1, 1, -1)$



Projection of $2\hat{i} - 2\hat{k}$ on $2\hat{i} + 3\hat{j} + \hat{k}$ is

$$= \frac{4-2}{\sqrt{4+9+1}} = \frac{2}{\sqrt{14}}$$

$$\therefore l^2 = 8 - \frac{4}{14} = \frac{108}{14}$$

$$\Rightarrow 14l^2 = 108$$

30. Let the complex numbers α and $\frac{1}{\alpha}$ lie on the circles $|z - z_0|^2 = 4$ and $|z - z_0|^2 = 16$ respectively, where $z_0 = 1 + i$. Then, the value of $100|\alpha|^2$ is.....

Ans. (20)

Sol. $|z - z_0|^2 = 4$

$$\Rightarrow (\alpha - z_0)(\bar{\alpha} - \bar{z}_0) = 4$$

$$\Rightarrow \alpha\bar{\alpha} - \alpha\bar{z}_0 - z_0\bar{\alpha} + |z_0|^2 = 4$$

$$\Rightarrow |\alpha|^2 - \alpha\bar{z}_0 - z_0\bar{\alpha} = 2 \dots\dots\dots(1)$$

$$|z - z_0|^2 = 16$$



$$\Rightarrow \left(\frac{1}{\alpha} - z_0\right)\left(\frac{1}{\alpha} - \bar{z}_0\right) = 16$$

$$\Rightarrow (1 - \alpha z_0)(1 - \alpha \bar{z}_0) = 16 |\alpha|^2$$

$$\Rightarrow 1 - \alpha z_0 - \alpha \bar{z}_0 + |\alpha|^2 |z_0|^2 = 16 |\alpha|^2$$

$$\Rightarrow 1 - \alpha z_0 - \alpha \bar{z}_0 = 14 |\alpha|^2 \dots\dots\dots(2)$$

From (1) and (2)

$$\Rightarrow 5 |\alpha|^2 = 1$$

$$\Rightarrow 100 |\alpha|^2 = 20$$

