FINAL JEE–MAIN EXAMINATION – JANUARY, 2024

(Held On Thursday 01st February, 2024) **TIME : 9 : 00 AM to 12 : 00 NOON**

MATHEMATICS — TEST PAPER WITH SOLUTION

SECTION-A

1. A bag contains 8 balls, whose colours are either white or black. 4 balls are drawn at random without replacement and it was found that 2 balls are white and other 2 balls are black. The probability that the bag contains equal number of white and black balls is:

(1) $\frac{2}{5}$ (2) $\frac{2}{7}$

(3) $\frac{1}{7}$ (4) $\frac{1}{5}$

Ans. (2)

Sol.

P(4W4B/2W2B) =

$$\frac{P(4W4B)\times P(2W2B/4W4B)}{P(2W6B)\times P(2W2B/2W6B)+P(3W5B)\times P(2W2B/3W5B)+\ldots\ldots\ldots\ldots+P(6W2B)\times P(2W2B/6W2B)}$$

$$=\frac{\frac{1}{5}\times\frac{{}^4C_2\times{}^4C_2}{{}^8C_4}}{\frac{1}{5}\times\frac{{}^2C_2\times{}^6C_2}{{}^8C_4}+\frac{1}{5}\times\frac{{}^3C_2\times{}^5C_2}{{}^8C_4}+\ldots+\frac{1}{5}\times\frac{{}^6C_2\times{}^2C_2}{{}^8C_4}}$$

$$=\frac{2}{7}$$

2. The value of the integral

$$\int_0^{\frac{\pi}{4}}\frac{xdx}{\sin^4(2x)+\cos^4(2x)}\ equals:$$

(1) $\frac{\sqrt{2}\pi^2}{8}$ (2) $\frac{\sqrt{2}\pi^2}{16}$

(3) $\frac{\sqrt{2}\pi^2}{32}$ (4) $\frac{\sqrt{2}\pi^2}{64}$

Ans. (3)

Sol. $\int_0^{\frac{\pi}{4}}\frac{xdx}{\sin^4(2x)+\cos^4(2x)}$

Let $2x=t$ then $dx=\frac{1}{2}dt$

$$I=\frac{1}{4}\int_0^{\frac{\pi}{2}}\frac{tdt}{\sin^4 t+\cos^4 t}$$

$$I=\frac{1}{4}\int_0^{\frac{\pi}{2}}\frac{\left(\frac{\pi}{2}-t\right)dt}{\sin^4\left(\frac{\pi}{2}-t\right)+\cos^4\left(\frac{\pi}{2}-t\right)}$$

$$I=\frac{1}{4}\int_0^{\frac{\pi}{2}}\frac{\frac{\pi}{2}dt}{\sin^4 t+\cos^4 t}-I$$

$$2I=\frac{\pi}{8}\int_0^{\frac{\pi}{2}}\frac{dt}{\sin^4 t+\cos^4 t}$$

$$2I=\frac{\pi}{8}\int_0^{\frac{\pi}{2}}\frac{\sec^4 tdt}{\tan^4 t+1}$$

Let tant = y then $\sec^2$t dt = dy

$$2I=\frac{\pi}{8}\int_0^{\infty}\frac{(1+y^2)dy}{1+y^4}$$

$$=\frac{\pi}{16}\int_0^{\infty}\frac{1+\frac{1}{y^2}}{y^2+\frac{1}{y^2}}dy$$

Put $y-\frac{1}{y}=p$

$$I=\frac{\pi}{16}\int_{-\infty}^{\infty}\frac{dp}{p^2+\left(\sqrt{2}\right)^2}$$

$$=\frac{\pi}{16\sqrt{2}}\left[\tan^{-1}\left(\frac{p}{\sqrt{2}}\right)\right]_{-\infty}^{\infty}$$

$$I=\frac{\pi^2}{16\sqrt{2}}$$

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3. If $A = \begin{bmatrix} \sqrt{2} & 1 \\ -1 & \sqrt{2} \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $C = ABA^T$ and $X = A^TC^2A$, then det X is equal to :

(1) 243

(2) 729

(3) 27

(4) 891

Ans. (2)

Sol.

$$A = \begin{bmatrix} \sqrt{2} & 1 \\ -1 & \sqrt{2} \end{bmatrix} \Rightarrow \det(A) = 3$$

$$B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \Rightarrow \det(B) = 1$$

Now $C = ABA^T \Rightarrow \det(C) = (\det(A))^2 \text{ x } \det(B)$

$|C| = 9$

Now $|X| = |A^TC^2A|$

$= |A^T|\,|C|^2\,|A|$

$= |A|^2\,|C|^2$

= 9 x 81

= 729

4. If $\tan A = \frac{1}{\sqrt{x(x^2+x+1)}}$, $\tan B = \frac{\sqrt{x}}{\sqrt{x^2+x+1}}$

and

$$\tan C = \left(x^{-3}+x^{-2}+x^{-1}\right)^{\frac{1}{2}}, 0 < A, B, C < \frac{\pi}{2}, then$$

A + B is equal to :

(1) C

(2) $\pi - C$

(3) $2\pi - C$

(4) $\frac{\pi}{2} - C$

Ans. (1)

Sol.

Finding tan (A + B) we get

$\Rightarrow$ tan (A + B) =

$$\frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\frac{1}{\sqrt{x(x^2+x+1)}} + \frac{\sqrt{x}}{\sqrt{x^2+x+1}}}{1 - \frac{1}{x^2+x+1}}$$

$$\Rightarrow \tan(A+B) = \frac{(1+x)\left(\sqrt{x^2+x+1}\right)}{\left(x^2+x\right)\left(\sqrt{x}\right)}$$

$$\frac{(1+x)\left(\sqrt{x^2+x+1}\right)}{\left(x^2+x\right)\left(\sqrt{x}\right)}$$

$$\tan(A+B) = \frac{\sqrt{x^2+x+1}}{x\sqrt{x}} = \tan C$$

$$A + B = C$$

5. If n is the number of ways five different employees can sit into four indistinguishable offices where any office may have any number of persons including zero, then n is equal to:

(1) 47

(2) 53

(3) 51

(4) 43

Ans. (3)

Sol.

Total ways to partition 5 into 4 parts are :

5, 0, 0, 0 $\Rightarrow$ 1 way

4, 1, 0, 0 $\Rightarrow \frac{5!}{4!} = 5$ ways

3, 2, 0, 0, $\Rightarrow \frac{5!}{3!2!} = 10$ ways

$2,2,0,1 \Rightarrow \frac{5!}{2!2!2!} = 15$ ways

$2,1,1,1 \Rightarrow \frac{5!}{2!(1!)^3 3!} = 10$ ways

$3,1,1,0 \Rightarrow \frac{5!}{3!2!} = 10$ ways

Total $\Rightarrow$ 1+5+10+15+10+10 = 51 ways

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6. Let $S=\{z\in C:|z-1|=1$ and $(\sqrt{2}-1)(z+\bar{z})-i(z-\bar{z})=2\sqrt{2}\}$. Let z_1, z_2 $\in S$ be such that $|z_1|=\max\limits_{z\in s}|z|$ and $|z_2|=\min\limits_{z\in s}|z|$.

Then $\left|\sqrt{2}z_1-z_2\right|^2$ equals :

(1) 1 (2) 4

(3) 3 (4) 2

Ans. (4)

Sol. Let Z = x + iy

Then $(x-1)^2+y^2=1 \quad \to (1)$

& $(\sqrt{2}-1)(2x)-i(2iy)=2\sqrt{2}$

$\Rightarrow (\sqrt{2}-1)x+y=\sqrt{2}\to(2)$

Solving (1) & (2) we get

Either x = 1 or $x=\dfrac{1}{2-\sqrt{2}}\to(3)$

On solving (3) with (2) we get

For x = 1 $\Rightarrow$ y = 1 $\Rightarrow$ $Z_2=1+i$

& for

$x=\dfrac{1}{2-\sqrt{2}}\Rightarrow y=\sqrt{2}-\dfrac{1}{\sqrt{2}}\Rightarrow Z_1=\left(1+\dfrac{1}{\sqrt{2}}\right)+\dfrac{i}{\sqrt{2}}$

Now

$\left|\sqrt{2}z_1-z_2\right|^2$

$=\left|\left(\dfrac{1}{\sqrt{2}}+1\right)\sqrt{2}+i-(1+i)\right|^2$

$=\left(\sqrt{2}\right)^2$

$=2$

7. Let the median and the mean deviation about the median of 7 observation 170, 125, 230, 190, 210, a, b be 170 and $\dfrac{205}{7}$ respectively. Then the mean deviation about the mean of these 7 observations is :

(1) 31

(2) 28

(3) 30

(4) 32

Ans. (3)

Sol. Median = 170 $\Rightarrow$125, a, b, 170, 190, 210, 230

Mean deviation about

Median =

$\dfrac{0+45+60+20+40+170-a+170-b}{7}=\dfrac{205}{7}$

$\Rightarrow$a + b = 300

Mean = $\dfrac{170+125+230+190+210+a+b}{7}=175$

Mean deviation

About mean =

$\dfrac{50+175-a+175-b+5+15+35+55}{7}=30$

8. Let $\vec{a}=-5\hat{i}+\hat{j}-3\hat{k},\vec{b}=\hat{i}+2\hat{j}-4\hat{k}$ and $\vec{c}=\left(\left(\left(\vec{a}\times\vec{b}\right)\times\hat{i}\right)\times\hat{i}\right)\times\hat{i}$. Then $\vec{c}\cdot\left(-\hat{i}+\hat{j}+\hat{k}\right)$ is equal to

(1) –12 (2) –10

(3) –13 (4) –15

Ans. (1)

Sol. $\vec{a}=-5\hat{i}+j-3\hat{k}$

$\vec{b}=\hat{i}+2\hat{j}-4\hat{k}$

$(\vec{a}\times\vec{b})\times\hat{i}=\left(\vec{a}\cdot\hat{i}\right)\vec{b}-\left(\vec{b}\cdot\hat{i}\right)\vec{a}$

$=-5\vec{b}-\vec{a}$

$=\left(\left(\left(-5\vec{b}-\vec{a}\right)\times\hat{i}\right)\times\hat{i}\right)$

$=\left(\left(-11\hat{j}+23\hat{k}\right)\times\hat{i}\right)\times\hat{i}$

$\Rightarrow\left(11\hat{k}+23\hat{j}\right)\times\hat{i}$

$\Rightarrow\left(11\hat{j}-23\hat{k}\right)$

$\vec{c}.\left(-\hat{i}+\hat{j}+\hat{k}\right)=11-23=-12$

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9. Let $S = \{x \in R : (\sqrt{3}+\sqrt{2})^x + (\sqrt{3}-\sqrt{2})^x = 10\}$.

Then the number of elements in S is :

(1) 4 (2) 0

(3) 2 (4) 1

Ans. (3)

Sol. $\left(\sqrt{3}+\sqrt{2}\right)^x + \left(\sqrt{3}-\sqrt{2}\right)^x = 10$

Let $\left(\sqrt{3}+\sqrt{2}\right)^x = t$

$t + \frac{1}{t} = 10$

$t^2 - 10t + 1 = 0$

$t = \frac{10 \pm \sqrt{100-4}}{2} = 5 \pm 2\sqrt{6}$

$\left(\sqrt{3}+\sqrt{2}\right)^x = \left(\sqrt{3} \pm \sqrt{2}\right)^2$

x = 2 or x = –2

Number of solutions = 2

10. The area enclosed by the curves xy + 4y = 16 and x + y = 6 is equal to :

(1) $28 - 30 \log_e 2$ (2) $30 - 28 \log_e 2$

(3) $30 - 32 \log_e 2$ (4) $32 - 30 \log_e 2$

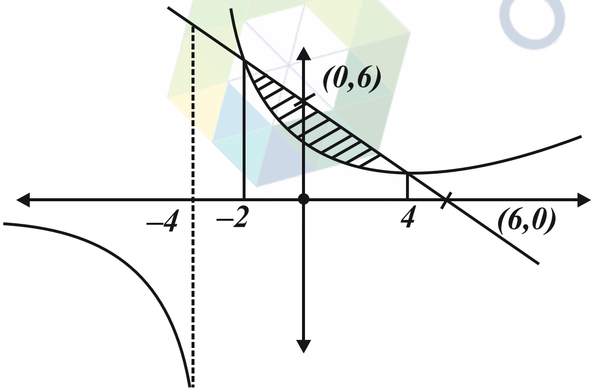
Ans. (3)

Sol. xy + 4y = 16 , x + y = 6

y(x + 4) = 16 ____(1) , x + y = 6___(2)

on solving, (1) & (2)

we get x = 4, x = –2



$$\text{Area} = \int_{-2}^{4} \left((6-x) - \left(\frac{16}{x+4} \right) \right) dx$$

$= 30 - 32 \ln 2$

11. Let $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ be defined as

$f(x) = \begin{cases} \log_e x & , \quad x > 0 \\ e^{-x} & , \quad x \le 0 \end{cases}$ and

$g(x) = \begin{cases} x & , \quad x \ge 0 \\ e^x & , \quad x < 0 \end{cases}$. Then, $gof : \mathbf{R} \to \mathbf{R}$ is :

(1) one-one but not onto

(2) neither one-one nor onto

(3) onto but not one-one

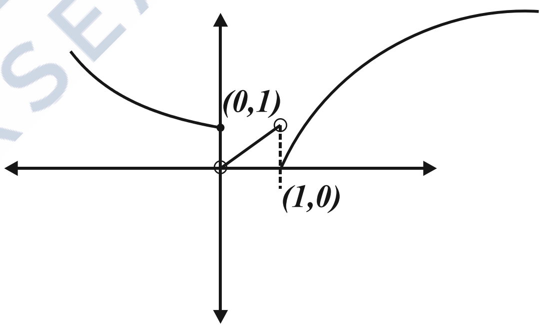
(4) both one-one and onto

Ans. (2)

Sol.

$g(f(x)) = \begin{cases} f(x), & f(x) \ge 0 \\ e^{f(x)}, & f(x) < 0 \end{cases}$

$g(f(x)) = \begin{cases} e^{-x}, & (-\infty, 0] \\ e^{\ln x}, & (0,1) \\ \ln x, & [1, \infty) \end{cases}$



Graph of g(f(x))

g(f(x)) ⟹ Many one into

12. If the system of equations

$2x + 3y - z = 5$

$x + \alpha y + 3z = -4$

$3x - y + \beta z = 7$

has infinitely many solutions, then $13\alpha\beta$ is equal to

(1) 1110 (2) 1120

(3) 1210 (4) 1220

Ans. (2)

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Sol. Using family of planes

$2x + 3y - z - 5 = k_1(x + \alpha y + 3z + 4) + k_2(3x - y + \beta z - 7)$

$2 = k_1 + 3k_2,\ 3 = k_1\alpha - k_2,\ -1 = 3k_1 + \beta k_2,\ -5 = 4k_1 - 7k_2$

On solving we get

$$k_2 = \frac{13}{19}, k_1 = \frac{-1}{19}, \alpha = -70, \beta = \frac{-16}{13}$$

$$13\alpha\beta = 13(-70)\left(\frac{-16}{13}\right) = 1120$$

13. For $0 < \theta < \pi/2$, if the eccentricity of the hyperbola $x^2 - y^2\operatorname{cosec}^2\theta = 5$ is $\sqrt{7}$ times eccentricity of the ellipse $x^2\operatorname{cosec}^2\theta + y^2 = 5$, then the value of θ is :

(1) $\frac{\pi}{6}$ (2) $\frac{5\pi}{12}$

(3) $\frac{\pi}{3}$ (4) $\frac{\pi}{4}$

Ans. (3)

Sol.

$$e_h = \sqrt{1+\sin^2\theta}$$

$$e_c = \sqrt{1-\sin^2\theta}$$

$$e_h = \sqrt{7}e_c$$

$$1+\sin^2\theta = 7(1-\sin^2\theta)$$

$$\sin^2\theta = \frac{6}{8} = \frac{3}{4}$$

$$\sin\theta = \frac{\sqrt{3}}{2}$$

$$\theta = \frac{\pi}{3}$$

14. Let $y = y(x)$ be the solution of the differential equation $\frac{dy}{dx} = 2x(x+y)^3 - x(x+y) - 1$, $y(0) = 1$.

Then, $\left(\frac{1}{\sqrt{2}} + y\left(\frac{1}{\sqrt{2}}\right)\right)^2$ equals :

(1) $\frac{4}{4+\sqrt{e}}$ (2) $\frac{3}{3-\sqrt{e}}$

(3) $\frac{2}{1+\sqrt{e}}$ (4) $\frac{1}{2-\sqrt{e}}$

Ans. (4)

Sol. $$\frac{dy}{dx} = 2x(x+y)^3 - x(x+y) - 1$$

$$x + y = t$$

$$\frac{dt}{dx} - 1 = 2xt^3 - xt - 1$$

$$\frac{dt}{2t^3 - t} = xdx$$

$$\frac{tdt}{2t^4 - t^2} = xdx$$

Let $t^2 = z$

$$\int \frac{dz}{2(2z^2 - z)} = \int xdx$$

$$\int \frac{dz}{4z\left(z - \frac{1}{2}\right)} = \int xdx$$

$$\ln\left|\frac{z - \frac{1}{2}}{z}\right| = x^2 + k$$

$$z = \frac{1}{2-\sqrt{e}}$$

15. Let $f : R \to R$ be defined as

$$f(x) = \begin{cases} \dfrac{a - b\cos 2x}{x^2} & ; \quad x < 0 \\ x^2 + cx + 2 & ; \quad 0 \le x \le 1 \\ 2x + 1 & ; \quad x > 1 \end{cases}$$

If f is continuous everywhere in **R** and m is the number of points where f is **NOT** differential then m + a + b + c equals :

(1) 1 (2) 4

(3) 3 (4) 2

Ans. (4)

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Sol. At x = 1, f(x) is continuous therefore,

$f(1^-) = f(1) = f(1^+)$

$f(1) = 3 + c$ ....(1)

$f(1^+) = \lim_{h\to0} 2(1 + h) + 1$

$f(1^+) = \lim_{h\to0} 3 + 2h = 3$ ....(2)

from (1) & (2)

c = 0

at x = 0, f(x) is continuous therefore,

$f(0^-) = f(0) = f(0^+)$ ....(3)

$f(0) = f(0^+) = 2$ ....(4)

$f(0^-)$ has to be equal to 2

$$\lim_{h\to0}\frac{a-b\cos(2h)}{h^2}$$

$$\lim_{h\to0}\frac{a-b\left\{1-\frac{4h^2}{2!}+\frac{16h^4}{4!}+\ldots\right\}}{h^2}$$

$$\lim_{h\to0}\frac{a-b+b\left\{2h^2-\frac{2}{3}h^4\ldots\right\}}{h^2}$$

for limit to exist a – b = 0 and limit is 2b ....(5)

from (3), (4) & (5)

a = b = 1

checking differentiability at x = 0

$$\text{LHD}: \lim_{h\to0}\frac{\frac{1-\cos 2h}{h^2}-2}{-h}$$

$$\lim_{h\to0}\frac{1-\left(1-\frac{4h^2}{2!}+\frac{16h^4}{4!}\ldots\right)-2h^2}{-h^3}=0$$

$$\text{RHD}: \lim_{h\to0}\frac{(0+h)^2+2-2}{h}=0$$

Function is differentiable at every point in its domain

$\therefore$ m = 0

m + a + b + c = 0 + 1 + 1 + 0 = 2

16. Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, a > b be an ellipse, whose eccentricity is $\frac{1}{\sqrt{2}}$ and the length of the latus rectum is $\sqrt{14}$. Then the square of the eccentricity of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is :

(1) 3 (2) 7/2

(3) 3/2 (4) 5/2

Ans. (3)

Sol.

$$e=\frac{1}{\sqrt{2}}=\sqrt{1-\frac{b^2}{a^2}}\Rightarrow\frac{1}{2}=1-\frac{b^2}{a^2}$$

$$\frac{2b^2}{a}=14$$

$$e_H=\sqrt{1+\frac{b^2}{a^2}}=\sqrt{1+\frac{1}{2}}=\sqrt{\frac{3}{2}}$$

$$(e_H)^2=\frac{3}{2}$$

17. Let 3, a, b, c be in A.P. and 3, a – 1, b + 1, c + 9 be in G.P. Then, the arithmetic mean of a, b and c is :

(1) –4 (2) –1

(3) 13 (4) 11

Ans. (4)

Sol.

3, a, b, c $\to$ A.P $\Rightarrow$ 3, 3+d, 3+2d, 3+3d

3, a–1, b+1, c + 9 $\to$ G.P $\Rightarrow$ 3, 2+d, 4+2d, 12+3d

a = 3 + d $\quad (2+d)^2=3(4+2d)$

b = 3 + 2d $\quad$ d = 4, -2

c = 3 + 3d

If d = 4 $\quad$ G.P $\Rightarrow$ 3, 6, 12, 24

a = 7

b = 11

c = 15

$$\frac{a+b+c}{3}=11$$

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18. Let C : $x^2 + y^2 = 4$ and C' : $x^2 + y^2 - 4\lambda x + 9 = 0$ be two circles. If the set of all values of λ so that the circles C and C' intersect at two distinct points, is **R**– [a, b], then the point (8a + 12, 16b – 20) lies on the curve :

(1) $x^2 + 2y^2 - 5x + 6y = 3$

(2) $5x^2 - y = -11$

(3) $x^2 - 4y^2 = 7$

(4) $6x^2 + y^2 = 42$

Ans. (4)

Sol. $x^2 + y^2 = 4$

C (0, 0) $r_1 = 2$

C' $(2\lambda, 0)$ $r_2 = \sqrt{4\lambda^2 - 9}$

$|r_1 - r_2| < CC' < |r_1 + r_2|$

$$\left|2-\sqrt{4\lambda^2-9}\right| < |2\lambda| < 2+\sqrt{4\lambda^2-9}$$

$$4 + 4\lambda^2 - 9 - 4\sqrt{4\lambda^2 - 9} < 4\lambda^2$$

True $\lambda \in$ R.... (1)

$$4\lambda^2 < 4 + 4\lambda^2 - 9 + 4\sqrt{4\lambda^2-9}$$

$5 < 4\sqrt{4\lambda^2 - 9}$ and $\lambda^2 \geq \frac{9}{4}$

$\frac{25}{16} < 4\lambda^2 - 9$ $\lambda \in \left(-\infty, -\frac{3}{2}\right] \cup \left[\frac{3}{2}, \infty\right)$

$\frac{169}{64} < \lambda^2$

$$\lambda \in \left(-\infty, -\frac{13}{8}\right) \cup \left(\frac{13}{8}, \infty\right) \quad \ldots(2)$$

from (1) and (2) $\lambda \in$

$$\lambda \in \left(-\infty, -\frac{13}{8}\right) \cup \left(\frac{13}{8}, \infty\right) \Rightarrow R - \left[-\frac{13}{8}, \frac{13}{8}\right]$$

as per question $a = -\frac{13}{8}$ and $b = \frac{13}{8}$

∴ required point is (–1, 6) with satisfies option (4)

19. If $5f(x) + 4f\left(\frac{1}{x}\right) = x^2 - 2, \forall\, x \neq 0$ and $y = 9x^2 f(x)$, then y is strictly increasing in :

(1) $\left(0, \frac{1}{\sqrt{5}}\right) \cup \left(\frac{1}{\sqrt{5}}, \infty\right)$

(2) $\left(-\frac{1}{\sqrt{5}}, 0\right) \cup \left(\frac{1}{\sqrt{5}}, \infty\right)$

(3) $\left(-\frac{1}{\sqrt{5}}, 0\right) \cup \left(0, \frac{1}{\sqrt{5}}\right)$

(4) $\left(-\infty, \frac{1}{\sqrt{5}}\right) \cup \left(0, \frac{1}{\sqrt{5}}\right)$

Ans. (2)

Sol. $5f(x) + 4f\left(\frac{1}{x}\right) = x^2 - 2, \ \forall x \neq 0 \ldots(1)$

Substitute $x \to \frac{1}{x}$

$$5f\left(\frac{1}{x}\right) + 4f(x) = \frac{1}{x^2} - 2 \quad \ldots(2)$$

On solving (1) and (2)

$$f(x) = \frac{5x^4 - 2x^2 - 4}{9x^2}$$

$y = 9x^2 f(x)$

$y = 5x^4 - 2x^2 - 4 \quad \ldots(3)$

$$\frac{dy}{dx} = 20x^3 - 4x$$

for strictly increasing

$$\frac{dy}{dx} > 0$$

$4x(5x^2 - 1) > 0$

$$x \in \left(-\frac{1}{\sqrt{5}}, 0\right) \cup \left(\frac{1}{\sqrt{5}}, \infty\right)$$

20. If the shortest distance between the lines $\frac{x-\lambda}{-2} = \frac{y-2}{1} = \frac{z-1}{1}$ and $\frac{x-\sqrt{3}}{1} = \frac{y-1}{-2} = \frac{z-2}{1}$ is 1, then the sum of all possible values of λ is :

(1) 0 (2) $2\sqrt{3}$

(3) $3\sqrt{3}$ (4) $-2\sqrt{3}$

Ans. (2)

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Sol. Passing points of lines L_1 & L_2 are

$$(\lambda, 2, 1) \& (\sqrt{3}, 1, 2)$$

$$S.D = \frac{\begin{vmatrix} \sqrt{3}-\lambda & -1 & 1 \\ -2 & 1 & 1 \\ 1 & -2 & 1 \end{vmatrix}}{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & 1 \\ 1 & -2 & 1 \end{vmatrix}}$$

$$1 = \left|\frac{\sqrt{3}-\lambda}{\sqrt{3}}\right|$$

$$\lambda = 0, \lambda = 2\sqrt{3}$$

SECTION-B

21. If x = x(t) is the solution of the differential equation $(t+1)dx = (2x + (t+1)^4)\,dt$, $x(0) = 2$, then, x(1) equals ___________.

Ans. (14)

Sol. $(t+1)dx = (2x + (t+1)^4)dt$

$$\frac{dx}{dt} = \frac{2x + (t+1)^4}{t+1}$$

$$\frac{dx}{dt} - \frac{2x}{t+1} = (t+1)^3$$

$$I \cdot F = e^{-\int \frac{2}{t+1}dt} = e^{-2\ln(t+1)} = \frac{1}{(t+1)^2}$$

$$\frac{x}{(t+1)^2} = \int \frac{1}{(t+1)^2}(t+1)^3 dt + c$$

$$\frac{x}{(t+1)^2} = \frac{(t+1)^2}{2} + c$$

$$\Rightarrow c = \frac{3}{2}$$

$$x = \frac{(t+1)^4}{2} + \frac{3}{2}(t+1)^2$$

put, t = 1

$x = 2^3 + 6 = 14$

22. The number of elements in the set

$S = \{(x, y, z) : x, y, z \in \mathbf{Z}, x + 2y + 3z = 42, x, y, z \geq 0\}$ equals __________.

Ans. (169)

Sol. $x + 2y + 3z = 42, \quad x, y, z \geq 0$

z = 0	$x + 2y = 42 \Rightarrow 22$
z = 1	$x + 2y = 39 \Rightarrow 20$
z = 2	$x + 2y = 36 \Rightarrow 19$
z = 3	$x + 2y = 33 \Rightarrow 17$
z = 4	$x + 2y = 30 \Rightarrow 16$
z = 5	$x + 2y = 27 \Rightarrow 14$
z = 6	$x + 2y = 24 \Rightarrow 13$
z = 7	$x + 2y = 21 \Rightarrow 11$
z = 8	$x + 2y = 18 \Rightarrow 10$
z = 9	$x + 2y = 15 \Rightarrow 8$
z = 10	$x + 2y = 12 \Rightarrow 7$
z = 11	$x + 2y = 9 \Rightarrow 5$
z = 12	$x + 2y = 6 \Rightarrow 4$
z = 13	$x + 2y = 3 \Rightarrow 2$
z = 14	$x + 2y = 0 \Rightarrow 1$

Total : 169

23. If the Coefficient of x^{30} in the expansion of $\left(1+\frac{1}{x}\right)^6 (1+x^2)^7 (1-x^3)^8$; $x \neq 0$ is α, then |α| equals __________.

Ans. (678)

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Sol. coeff of x^{30} in $\dfrac{(x+1)^6(1+x^2)^7(1-x^3)^8}{x^6}$

coeff. of x^{36} in $(1+x)^6(1+x^2)^7(1-x^3)^8$

General term

$^6C_{r_1}\,{}^7C_{r_2}\,{}^8C_{r_3}(-1)^{r_3}x^{r_1+2r_2+3r_3}$

$r_1+2r_2+3r_3=36$

Case-I : $r_1+2r_2=12$ (Taking $r_3 = 8$)

r_1	r_2	r_3
0	6	8
2	5	8
4	4	8
6	3	8

Case-II : $r_1+2r_2=15$ (Taking $r_3 = 7$)

r_1	r_2	r_3
1	7	7
3	6	7
5	5	7

Case-III : $r_1+2r_2=18$ (Taking $r_3 = 6$)

r_1	r_2	r_3
4	7	6
6	6	6

Coeff. = 7 + (15 × 21) + (15 × 35) + (35)

–(6 × 8) – (20 × 7 × 8) – (6 × 21 × 8) + (15 × 28)

+ (7 × 28) = –678 = α

|α| = 678

24. Let 3, 7, 11, 15, ..., 403 and 2, 5, 8, 11, ..., 404 be two arithmetic progressions. Then the sum, of the common terms in them, is equal to _________.

Ans. (6699)

Sol. 3, 7, 11, 15, ….., 403

2, 5, 8, 11, …., 404

LCM (4, 3) = 12

11, 23, 35,….. let (403)

403 = 11 + (n - 1) x 12

$\dfrac{392}{12}=n-1$

$33\cdot 66=n$

$n=33$

Sum $\dfrac{33}{2}(22+32\times 12)$

=6699

25. Let {x} denote the fractional part of x and $f(x) = \dfrac{\cos^{-1}(1-\{x\}^2)\sin^{-1}(1-\{x\})}{\{x\}-\{x\}^3}$, $x \neq 0$. If L and R respectively denotes the left hand limit and the right hand limit of f(x) at x = 0, then $\dfrac{32}{\pi^2}(L^2+R^2)$ is equal to ____________.

Ans. (18)

Sol. Finding right hand limit

$\lim\limits_{x\to 0^+} f(x)=\lim\limits_{h\to 0} f(0+h)$

$=\lim\limits_{h\to 0} f(h)$

$=\lim\limits_{h\to 0}\dfrac{\cos^{-1}(1-h^2)\sin^{-1}(1-h)}{h(1-h^2)}$

$=\lim\limits_{h\to 0}\dfrac{\cos^{-1}(1-h^2)}{h}\left(\dfrac{\sin^{-1}1}{1}\right)$

Let $\cos^{-1}(1-h^2)=\theta \Rightarrow \cos\theta = 1-h^2$

$=\dfrac{\pi}{2}\lim\limits_{\theta\to 0}\dfrac{\theta}{\sqrt{1-\cos\theta}}$

$=\dfrac{\pi}{2}\lim\limits_{\theta\to 0}\dfrac{1}{\sqrt{\dfrac{1-\cos\theta}{\theta^2}}}$

$=\dfrac{\pi}{2}\dfrac{1}{\sqrt{1/2}}$

$R=\dfrac{\pi}{\sqrt{2}}$

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Now finding left hand limit

$$L=\lim_{x\to 0^-} f(x)$$

$$=\lim_{h\to 0} f(-h)$$

$$=\lim_{h\to 0}\frac{\cos^{-1}\left(1-\{-h\}^2\right)\sin^{-1}\left(1-\{-h\}\right)}{\{-h\}-\{-h\}^3}$$

$$=\lim_{h\to 0}\frac{\cos^{-1}\left(1-(-h+1)^2\right)\sin^{-1}\left(1-(-h+1)\right)}{(-h+1)-(-h+1)^3}$$

$$=\lim_{h\to 0}\frac{\cos^{-1}\left(-h^2+2h\right)\sin^{-1}h}{(1-h)\left(1-(1-h)^2\right)}$$

$$=\lim_{h\to 0}\left(\frac{\pi}{2}\right)\frac{\sin^{-1}h}{\left(1-(1-h)^2\right)}$$

$$=\frac{\pi}{2}\lim_{h\to 0}\left(\frac{\sin^{-1}h}{-h^2+2h}\right)$$

$$=\frac{\pi}{2}\lim_{h\to 0}\left(\frac{\sin^{-1}h}{h}\right)\left(\frac{1}{-h+2}\right)$$

$$L=\frac{\pi}{4}$$

$$\frac{32}{\pi^2}\left(L^2+R^2\right)=\frac{32}{\pi^2}\left(\frac{\pi^2}{2}+\frac{\pi^2}{16}\right)$$

$= 16 + 2$

$= 18$

26. Let the line L : $\sqrt{2}\,x + y = \alpha$ pass through the point of the intersection P (in the first quadrant) of the circle $x^2 + y^2 = 3$ and the parabola $x^2 = 2y$. Let the line L touch two circles C_1 and C_2 of equal radius $2\sqrt{3}$. If the centres Q_1 and Q_2 of the circles C_1 and C_2 lie on the y-axis, then the square of the area of the triangle PQ_1Q_2 is equal to ___________.

Ans. (72)

Sol. $x^2 + y^2 = 3$ and $x^2 = 2y$

$y^2 + 2y - 3 = 0 \Rightarrow (y + 3)(y - 1) = 0$

$y = -3$ or $y = 1$

$y = 1$ $x = \sqrt{2} \Rightarrow P(\sqrt{2},1)$

p lies on the line

$\sqrt{2}x+y=\alpha$

$\sqrt{2}(\sqrt{2})+1=\alpha$

$\alpha=3$

For circle C_1

Q_1 lies on y axis

Let Q_1 $(0,\alpha)$ coordinates

$R_1 = 2\sqrt{3}$ (Given

Line L act as tangent

Apply P = r (condition of tangency)

$$\Rightarrow\left|\frac{\alpha-3}{\sqrt{3}}\right|=2\sqrt{3}$$

$$\Rightarrow|\alpha-3|=6$$

$\alpha-3=6$ or $\alpha-3=-6$

$\Rightarrow \alpha=9$ $\quad \alpha=-3$

$$\triangle PQ_1Q_2=\frac{1}{2}\begin{vmatrix}\sqrt{2} & 1 & 1\\ 0 & 9 & 1\\ 0 & -3 & 1\end{vmatrix}$$

$$=\frac{1}{2}\left(\sqrt{2}(12)\right)=6\sqrt{2}$$

$$\left(\triangle PQ_1Q_2\right)^2=72$$

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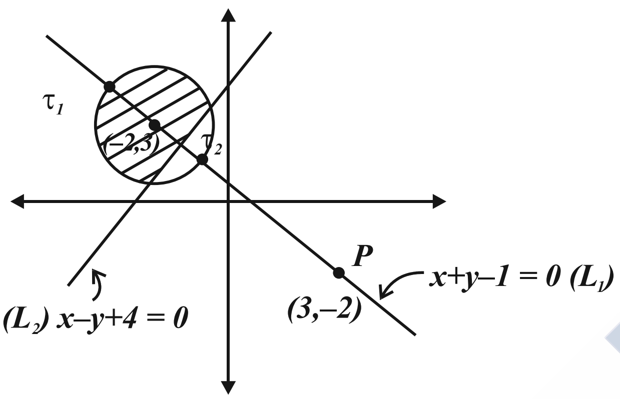
27. Let P = {z ∈ ℂ : |z + 2 – 3i | ≤ 1} and Q = {z ∈ ℂ : z (1 + i) + $\bar{z}$ (1 – i) ≤ –8} . Let in P ⋂ Q, |z – 3 + 2i| be maximum and minimum at z_1 and z_2 respectively. If $|z_1|^2 + 2|z|^2 = \alpha + \beta\sqrt{2}$, where α, β are integers, then α + β equals ________.

Ans. (36)

Sol.



Clearly for the shaded region z_1 is the intersection of the circle and the line passing through P (L_1) and z_2 is intersection of line L_1 & L_2

Circle : $(x + 2)^2 + (y - 3)^2 = 1$

$L_1 : x + y – 1 = 0$

$L_2 : x – y + 4 = 0$

On solving circle & L_1 we get

$z_1 : \left(-2-\frac{1}{\sqrt{2}}, 3+\frac{1}{\sqrt{2}}\right)$

On solving L_1 and z_2 is intersection of line L_1 & L_2

we get $z_2 : \left(\frac{-3}{2}, \frac{5}{2}\right)$

$$|z_1|^2 + 2|z_2|^2 = 14 + 5\sqrt{2} + 17$$

$$= 31 + 5\sqrt{2}$$

So $\alpha = 31$

$\beta = 5$

$\alpha + \beta = 36$

28. If $\int_{-\pi/2}^{\pi/2} \frac{8\sqrt{2}\cos x dx}{(1+e^{\sin x})(1+\sin^4 x)} = \alpha\pi + \beta \log_e (3 + 2\sqrt{2})$, where α, β are integers, then $\alpha^2 + \beta^2$ equals ________.

Ans. (8)

Sol. $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{8\sqrt{2}\cos x}{\left(1+e^{\sin x}\right)\left(1+\sin^4 x\right)} dx$

Apply king

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{8\sqrt{2}\cos x\left(e^{\sin x}\right)}{\left(1+e^{\sin x}\right)\left(1+\sin^4 x\right)} dx \quad ....(2)$$

adding (1) & (2)

$$2I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{8\sqrt{2}\cos x}{1+\sin^4 x} dx$$

$$I = \int_0^{\frac{\pi}{2}} \frac{8\sqrt{2}\cos x}{1+\sin^4 x} dx,$$

$\sin x = t$

$$I = \int_0^1 \frac{8\sqrt{2}}{1+t^4} dx$$

$$I = 4\sqrt{2}\int_0^1 \left(\frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}} - \frac{1-\frac{1}{t^2}}{t^2+\frac{1}{t^2}} \right) dt$$

$$I = 4\sqrt{2}\int_0^1 \frac{\left(1+\frac{1}{t^2}\right)}{\left(t-\frac{1}{t}\right)^2+2} - \frac{\left(1-\frac{1}{t^2}\right)}{\left(t+\frac{1}{t}\right)^2-2} dt$$

$$Let\ t - \frac{1}{t} = z \ \&\ t + \frac{1}{t} = k$$

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$$=4\sqrt{2}\left[\int_{-\infty}^{0}\frac{dz}{z^2+2}-\int_{\infty}^{2}\frac{dk}{k^2-2}\right]$$

$$=4\sqrt{2}\left[\frac{1}{\sqrt{2}}\tan^{-1}\frac{z}{\sqrt{2}}\right]_{-\infty}^{0}-\left[\frac{1}{2\sqrt{2}}\ln\left(\frac{k-\sqrt{2}}{k+\sqrt{2}}\right)\right]_{\infty}^{2}$$

$$=4\sqrt{2}\left[\frac{\pi}{2\sqrt{2}}-\frac{1}{2\sqrt{2}}\left[\ln\frac{2-\sqrt{2}}{2+\sqrt{2}}\right]\right]$$

$$=2\pi+2\ln(3+2\sqrt{2})$$

$\alpha=2$

$\beta=2$

29. Let the line of the shortest distance between the lines

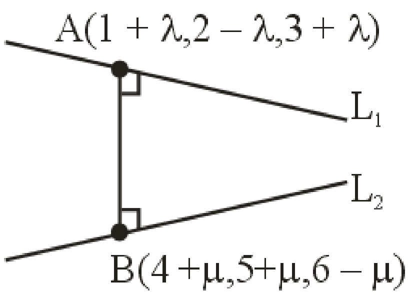
$L_1 : \vec{r}=\left(\hat{i}+2\hat{j}+3\hat{k}\right)+\lambda\left(\hat{i}-\hat{j}+\hat{k}\right)$ and

$L_2 : \vec{r}=\left(4\hat{i}+5\hat{j}+6\hat{k}\right)+\mu\left(\hat{i}+\hat{j}-\hat{k}\right)$

intersect L_1 and L_2 at P and Q respectively. If (α, β, γ) is the midpoint of the line segment PQ, then $2(\alpha+\beta+\gamma)$ is equal to ____________.

Ans. (21)

Sol.



$\vec{b}=\hat{i}-\hat{j}+\hat{k}$ (DR's of L_1)

$\vec{d}=\hat{i}+\hat{j}-\hat{k}$ (DR's of L_2)

$$\vec{b}\times\vec{d}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 1\\ 1 & 1 & -1\end{vmatrix}$$

$=0\hat{i}+2\hat{j}+2\hat{k}$ (DR's of Line perpendicular to L_1 and L_2)

DR of AB line

$=(0,2,2)=(3+\mu-\lambda,3+\mu+\lambda,3-\mu-\lambda)$

$$\frac{3+\mu-\lambda}{0}=\frac{3+\mu+\lambda}{2}=\frac{3-\mu-\lambda}{2}$$

Solving above equation we get $\mu=-\frac{3}{2}$ and $\lambda=\frac{3}{2}$

point $A=\left(\frac{5}{2},\frac{1}{2},\frac{9}{2}\right)$

$B=\left(\frac{5}{2},\frac{7}{2},\frac{15}{2}\right)$

Point of AB $=\left(\frac{5}{2},2,6\right)=(\alpha,\beta,\gamma)$

$2(\alpha+\beta+\gamma)=5+4+12=21$

30. Let A= {1, 2, 3, . . 20}. Let R_1 and R_2 two relation on A such that

R_1 = {(a, b) : b is divisible by a}

R_2 = {(a, b) : a is an integral multiple of b}.

Then, number of elements in $R_1 - R_2$ is equal to _________.

Ans. (46)

Sol. $n(R_1)=20+10+6+5+4+3+2+2+2$

$+2+\underbrace{1+\ldots+1}_{10\text{ times}}$

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$n(R_1) = 66$

$R_1 \cap R_2 = \{(1,1),(2,2),...(20,20)\}$

$n(R_1 \cap R_2) = 20$

$n(R_1 - R_2) = n(R_1) - n(R_1 \cap R_2)$

$= n(R_1) - 20$

$= 66 - 20$

$R_1 - R_2 = 46$ Pair

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